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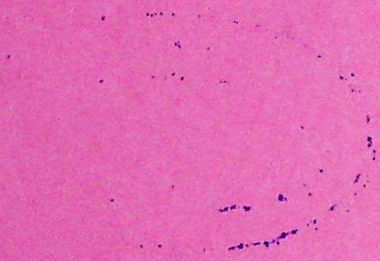

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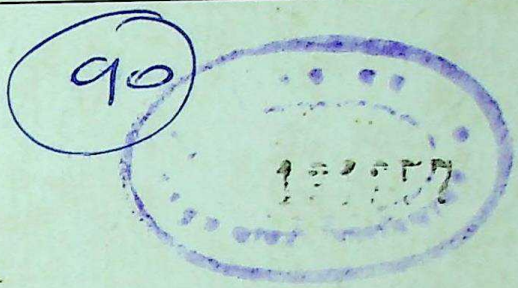

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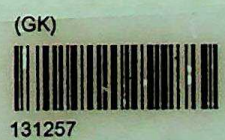

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CONTENTS - SADRŽAJ

BOGOLJUB STANKOVIĆ	ON THE IMPOSSIBILITY OF EXTENDING THE PRODUCT OF CONTINUOUS FUNCTIONS TO THE FIELD OF MIKUSIŃSKI OPERATORS	1
OLGA HADŽIĆ	ON COMMON FIXED POINT THEOREMS IN 2-METRIC SPACES	7
OLGA HADŽIĆ	A LERAY-SCHAUDER PRINCIPLE FOR MULTI-VALUED MAPPINGS IN TOPOLOGICAL VECTOR SPACES	19
OLGA HADŽIĆ, LJILJANA GAJIĆ	ON THE EQUATION $Tx=F(x,Q(Tx))$ IN LO - CALLY CONVEX SPACES	31
DO HONG TAN	A FIXED POINT THEOREM FOR MULTIVALUED QUASI-CONTRACTIONS IN PROBABILISTIC METRIC SPACES	43
MILA STOJAKOVIĆ	A FIXED POINT THEOREM IN PROBABILISTIC METRIC SPACES	55
ILIJA KOVAČEVIĆ	ON MULTIFUNCTIONS AND PARACOMPACTNESS..	61
MIRKO BUDINČEVIĆ	ON WONG 'S CONJUCTURE	69
MIRKO BUDINČEVIĆ M. R. S. KULENOVIĆ	ASYMPTOTIC ANALYSIS OF A NONLINEAR SECOND ORDER DIFFERENCE EQUATION II	73
STEVAN PILIPOVIĆ	AN ANALYTIC CHARACTERIZATION OF GENERALIZED FUNCTIONS WHICH HAVE A LAGUERRE EXPANSION	93
ZORAN IVKOVIĆ, ZAGORKA LOZANOV	ON HERMITE POLYNOMIALS OF THE GAUSSIAN RANDOM PROCESS	105
ZORAN IVKOVIĆ	ON SPECTRAL TYPE OF NONLINEAR AND NONANTICIPATIVE TRANSFORMATION OF THE WIENER PROCESS	117

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DRAGOSLAV HERCEG	SOME DIFFERENCE SCHEMES FOR TWO POINT BOUNDARY VALUED PROBLEMS	123
DRAGOSLAV HERCEG	EXIT CRITERIA FOR SOME ITERATIVE METHODS	139
JENS LORENZ	STABILITY AND MONOTONICITY PROPERTIES OF STIFF QUASILINEAR BOUNDARY PROBLEMS...	151
LJILJANA CVETKOVIĆ, DRAGOSLAV HERCEG	ON A NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEM USING AN OPTIMAL NUMERICAL DIFFERENTIATION	177
KATARINA SURLA	ON THE CONVERGENCE OF SOME FINITE DIFFERENCE SCHEMES FOR A SINGULAR PERTURBATION PROBLEM	191
RELJA VULANOVIĆ	AN EXPONENTIALLY FITTED SCHEMES FOR A SINGULAR PERTURBATION PROBLEM	205
VOJISLAV STOJKOVIĆ	SOME PROPERTIES OF SYNTAX-DIRECTED TRANSLATIONS	217
IVAN STOJMENOVIĆ, VOJISLAV STOJKOVIĆ	SOME PROOFS BY STRUCTURAL INDUCTION ON LISTS	235
DĂNUŢ MARCU	SOME ALGEBRAIC PROPERTIES OF REGULAR MATROIDS	251
DRAGAN ACKETA	ANOTHER CONSTRUCTION OF RANK 4 PAVING MATROIDS ON 8 ELEMENTS (I)	259
DRAGAN ACKETA	ANOTHER CONSTRUCTION OF RANK 4 PAVING MATROIDS ON 8 ELEMENTS (II)	277
RATKO TOŠIĆ	A SEQUENTIAL WEIGHING PROCEDURE	305
RATKO TOŠIĆ VOJISLAV PETROVIĆ	COUNTING BINARY GRIDS	311

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RATKO TOŠIĆ, RADE DOROSLOVAČKI	A COMBINATORIAL IDENTITY AND ITS APPLICATIONS	319
MILAN GRULOVIĆ	A NOTE ON FORCING AND WEAK INTERPOLATION THEOREM FOR INFINITARY LOGICS	327
MILEVA PRVANOVIĆ	A GENERALIZATION OF THE BOCHNER AND CONSTANT BOCHNER CURVATURE TENSOR ..	349
JOVANKA NIKIĆ	ON A STRUCTURE DEFINED BY A TENSOR FIELD f OF THE TYPE (1,1) SATISFYING $f^{2\cdot 2^q+1} - f = 0$	369
NEVENA PUŠIĆ	WEYL-OTSUKI SPACES OF THE SECOND KIND WITH A SPECIAL TENSOR P	379
ЯНЕЗ УШАН	О ОДНОМ КЛАССЕ КОНЕЧНЫХ к-ПОЛУСЕТЕЙ	387
ZORAN STOJAKOVIĆ	ON BISYMMETRIC [n,m]-GROUPOIDS ..	399
ZORAN STOJAKOVIĆ	CYCLIC n-QUASIGROUPS	407
STOJAN BOGDANOVIĆ	POWER REGULAR SEMIGROUP	417
STOJAN BOGDANOVIĆ SILVIJA GILEZAN	SEMIGROUPS WITH COMPLETELY SIMPLE KERNEL	429
BRANIMIR ŠEŠELJA	FUZZY CONGRUENCE RELATIONS AND CONSTRUCTIONS OF ALGEBRAS	447

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ON THE IMPOSSIBILITY OF EXTENDING THE PRODUCT OF CONTINUOUS FUNCTIONS TO THE FIELD OF MIKUSIŃSKI OPERATORS

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ABSTRACT

Our aim is to show that the usual product of continuous functions can not be extended on the whole field M (the field of Mikusiński operators) or on some special subsets of M, as an inner operation, if we require the extended operation to preserve some natural properties of the product of continuous functions.

INTRODUCTION

The field M of Mikusiński operators [1] was constructed by the extension of the ring C of continuous functions $f=\{f(t)\}$ defined on the interval $[0,\infty)$ and provided with two operations: sum and convolution $(fg = \int_0^t f(t-u)\cdot g(u)du)$. In C we have the countable and saturated family of semi-norms: $\|f\|_n = \max_{0<t\leq n}|f(t)|$ and in this way a topology. In M we define a convergence class too. Usually we work with the following one (called type I) : the sequence $\{a_n\}\subset M$ converges in M to a if and only if there exists $q\in M$ such that $qa_n\in C$ and qa_n converges to qa in C. This class is not topological.

The field M contains the ring L of local integrable functions defined on the interval $[0,\infty)$.

The default of an inner operation in M (let us denote it by ..), with the restriction on C as the usual product, diminishes the possibility of applying operators, especially in the theory of differential equations. This is the reason to ask the following question: Is it possible to define in M such an

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operation $\cdot\cdot$ which preserves some natural properties of the product of continuous functions? The answer to this question is the subject of discussion of this paper.

1. EXTENSION OF THE CLASSICAL DERIVATIVE TO THE ELEMENTS OF C

In the field M, the element ℓ (the integral operator which is from C) has, in M, the inverse element s (which is not from C). Using the operator s, we can enlarge the classical derivative to every continuous function $f \in C$: $Df = sf - f(0)I$; I is the unit element in the field M. This derivative D has the two following properties:

- If the function $f \in C$ has the usual derivative $f' \in L$, then $Df = f'$. This follows from the fact that in this case $sf = \{f'(t)\} + f(0)\, I$.

- If the functions f and g from C have as their derivatives f' and g' belonging to L, then in M there is:

$$(1) \qquad D\{f(t)\cdot g(t)\} = \{g(t)\cdot Df(t)\} + \{f(t)\cdot Dg(t)\} .$$

2. THE SET P

By P we denote every subset from M which contains at least the ring C and elements of the form Df, $f \in C$. Let us suppose that in P is defined an inner operation $\cdot\cdot$ which is associative, and has the unit element $\{1\}$ and the restriction of which on C is the usual product. For this new operation we suppose something more: If f and g belong to C and their derivatives belong to L, then:

$$(2) \qquad D(f \cdot\cdot g) = Df \cdot\cdot g + f \cdot\cdot Dg .$$

In P we induce the convergence class from M.

PROPOSITION. *The mapping* P *into* $P : a \to a_0 \cdot\cdot a$, *where* a_0 *is any element from* P, *is not sequentially continuous.*

The proof of this proposition is based on two lemmas.

LEMMA 1. *The inverse element of* $\{qt^p\} \in C$, $q \neq 0$, $0 < p < 1$, *exists in* P *for the operation* $\cdot\cdot$ *and it is just* $D\{rt^{1-p}\}$, $q.r = \frac{1}{1-p}$.

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P r o o f of lemma 1. - Let h be an element of P such that:

$$D\{r\ t^{1-p}\} \cdot\cdot \{q\ t^{p}\} = h\ .$$

Because of the associativity of the operation .. :

$$D\{r\ t^{1-p}\} \cdot\cdot \{q\ t^{p+1}\} = h \cdot\cdot \{t\}\ .$$

Using relation (2) and the properties of D we have:

$$D\left[\{rt^{1-p}\} \cdot\cdot \{qt^{p+1}\}\right] - \left[\{rt^{1-p}\} \cdot\cdot D\{qt^{p+1}\}\right] = h \cdot\cdot \{t\}\ ,$$

respectively:

$\{t\} = h \cdot\cdot \{t\}$, which gives $h = \{1\}$.

In a like manner we can prove that:

$$\{qt^{p}\} \cdot\cdot D\{rt^{1-p}\} = \{1\}\ .$$

LEMMA 2. $\lim\limits_{x\to o^+} \Gamma(x)\ell^{x}$, $x > 0$, *does not exist in* M .

P r o o f of lemma 2. - Let us suppose that such a limit exists; then there exists a $q \in M$ such that $\Gamma(x)\ell^{x}q$ belongs to C and converges in C. From the continuity of the second operation in M it follows that there exists an element $w \in C$ such that $\pm\Gamma(x)\ell^{2+x}\Gamma(2+x)w$ belongs to C converges in C too.

We know that $\int_o^t u^{1+x}\cdot w(t-u)du$ converges uniformly to $\int_o^t u\cdot w(t-u)du$ in every interval $[0,T]$, $T < \infty$, as $x \to 0_+$. Because of the Titchmarsh theorem [2] there exists $t_o \in (0,T_o)$ such that $\int_o^{t_o} u\cdot w(t_o-u)du = r \neq 0$. We can suppose that $r > 0$ (without any restriction). Then we also have an interval $[t_o-\eta,\ t_o+\eta] \subset [0,T_o]$ such that the continuous function $\int_o^t u\cdot w(t-u)du > r' > 0$. Let us chosse an ε in such a way that $0 < \varepsilon < r'$. For this ε there exists $\delta(\varepsilon)$ such that:

$$\int_o^t u^{1+x}w(t-u)du \gtrless \int_o^t u\cdot w(t-u)du - \varepsilon, \quad 0 < x < \delta(\varepsilon)\ ,$$

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for every $t \in [0, T_o]$ and also for $t \in [t_o-\eta, t_o+\eta]$.

We know that $\Gamma(x) > 0$, $x > 0$ and $\Gamma(x) \to \infty$ for $x \to 0$, $x > 0$. Consequently:

$$\Gamma(x) \int_o^t u^{1+x} \cdot w(t-u)du \geq (r'-\varepsilon)\Gamma(x)$$

for $t \in [t_o-\eta, t_o+\eta]$ and $\Gamma(x)\Gamma(2+x)\ell^{2+x}w$ cannot converge in C.

P r o o f of the proposition. - Let us start from the sequence $\{a_n\} \subset P$:

$$a_n = \{t^v\} \cdot\cdot D\{nt^{1/n}\}, \quad 0 < v < 1 .$$

First we shall find an other analytical expression for it. For this aim let us "multiply" a_n by

$$\{\frac{1}{n-1} t^{1-v-1/n}\} \in C, \quad n \geq n_o > 1, \quad 1-v-1/n_o > 0 :$$

$$\{\frac{1}{n-1} t^{1-v-1/n}\} \cdot\cdot a_n = \{\frac{t^{1-1/n}}{n-1}\} \cdot\cdot D\{nt^{1/n}\} =$$

$$= D\{\frac{n}{n-1} t\} \cdot\cdot D\{\frac{t^{1-1/n}}{n-1}\} \cdot\cdot \{nt^{1/n}\}.$$

By lemma 1 we have:

$$= \{\frac{n}{n-1}\} - \{1\} = \{\frac{1}{n-1}\} .$$

Now we can conclude that:

$$\{t^{1-v-1/n}\} \cdot\cdot a_n = \{1\}$$

and a_n, $n \geq n_o > 1$, $1-v-1/n_o > 0$ is the inverse element to $\{t^{1-v-1/n}\}$. Lemma 1 says that:

$$a_n = D\{\frac{t^{v+1/n}}{v+1/n}\} = \Gamma(v+1/n)\ell^{v+1/n} .$$

Now we shall prove that a_n converges in P to $D\{\frac{t^v}{v}\}$. a_n belongs to C and converges in C to $\Gamma(v)\ell^{1+v} = \ell\,\Gamma(v)\ell^v =$ $= \ell D\{\frac{t^v}{v}\}$.

The element $D\{\frac{t^{1-v}}{1-v}\} \in P$ can be our a_o from the proposition. "Multiplying" a_n by the so defined a_o and taking care of lemma 1 we have:

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$$D\{\frac{t^{1-v}}{1-v}\} \cdot\cdot \{t^{v}\} \cdot\cdot D\{nt^{1/n}\} = D\{nt^{1/n}\} = \Gamma(1/n)\ell^{1/n}$$

and this sequence does not converge in P (by lemma 2).

CONSEQUENCE OF THE PROPOSITION. The consequence of our proposition is that the usual product of continuous functions cannot be extended to a set of the form P (as well as M) as an inner operation, if we require of the extended operation: to be associative; to have the unit element $|1|$; for $f,g \in C$ with derivatives $f',g' \in L$, the relation (2) to be satisfied and the application $a \to a_0 \cdot\cdot a$, P into P, to be sequentially continuous for every $a_0 \in P$.

REMARK. Our proposition remains true if we change the convergence class in M and consequently in P under only one condition: that $\Gamma(x)\ell^{x}$ does not converge when $x \to 0_+$, in this new convergence class. This is the case if we replace the defined convergence class by the so-called convergence class type II.

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REZIME

O NEMOGUĆNOSTI PROŠIRENJA NEPREKIDNIH FUNKCIJA NA POLJE OPERATORA MIKUSIŃSKOG

Obeležićemo sa C prsten neprekidnih funkcija, a sa L prsten lokalno integrabilnih funkcija definisanih nad $[0,\infty)$. M je polje operatora Mikusińskog |1|, a D operacija koja presli

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kava C u M i koja uopštava klasični izvod: $Df = sf - f(0)\,I$ gde su s i I elementi iz M (s operator diferenciranja, a I jedinični elemenat). P će biti svaki podskup od M koji sadrži bar C i elemente oblika Df, $f \in C$.

Pokazano je da se u P (pa time i u M) ne može definisati unutrašnja operacija $\cdot\cdot$ ako se od nje zahteva da je njena restrikcija nad C obično množenje, da je asocijativna, da ima jedinični elemenat {1}, da za f i g iz C koje imaju izvode f´ i g´ iz L bude zadovoljena relacija (2) i da je preslikavanje $a \to a_o \cdot\cdot$ a sekvencijalno neprekidno za svako $a_o \in P$. Sekvencijalna neprekidnost se odnosi na konvergentnu klasu tipa I i II kao i svaku drugu u kojoj ne postoji $\lim_{x \to o^+} \Gamma(x)\ell^x$.

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ON COMMON FIXED POINT THEOREMS IN 2-METRIC SPACES

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ABSTRACT

In |3| S.Gähler introduced the notion of 2-metric space and many fixed point theorems in such spaces are proved in |8|, |10|, |11|, |12|, |13|, |14|, |15|.

Here we shall prove a result similar to Theorem 2 |7| for bounded and complete 2-metric spaces. Also two common fixed point theorems for a family $\{A_j\}_{j\in N}$, S and T are obtained.

First, we shall give some definitions from |3|. Let X be an arbitrary set and d a real valued function on $X \times X \times X$ satisfying the following conditions:

1. To each pair of point $(x,y) \in X \times X$ with $x \neq y$ there is $z \in X$ such that $d(x,y,z) \neq 0$.
2. $d(x,y,z) = 0$ only when at least two of tree points are equal.
3. For every $x,y,z \in X$:
$$d(x,y,z) = d(x,z,y) = d(y,z,x).$$
4. For every $x,y,z,u \in X$:
$$d(x,y,z) \leq d(x,y,u) + d(x,u,z) + d(u,y,z).$$

Then (X,d) is said to be a 2-metric space. It is easy to see that d is non-negative functional.

The convergence in a 2-metric space (X,d) is introduced in the following way |3|.

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Let $\{x_n\}_{n\in N}$ be a sequence from X and $x \in X$. We say that the sequence $\{x_n\}_{n\in N}$ converges to x if and only if for every $a \in X$, $\lim_{n\to\infty} d(x_n,x,a) = 0$. The sequence $\{x_n\}_{n\in N}$ is a Cauchy sequence if $\lim_{m,n\to\infty} d(x_m,x_n,a) = 0$, for all $a \in X$. If every Cauchy sequence in X converges to a point in X we say that (X,d) is a complete 2-metric space.

THEOREM 1. *Let* (X,d) *be a bounded, complete* 2*-metric space,* S *and* T *be one to one continuous mapping from* X *into* X , $A : X \to SX \cap TX$ *be continuous and* T *and* S *be commutative with* A. *If for every* $x \in X$ *there exists* $n(x) \in N$ *so that for every* $y \in X$ *and every* $a \in X$*:*

$$d(A^{n(x)}x, A^{n(x)}y, a) \leq q \min\{d(Sx,Ty,a), d(Tx,Sy,a)\}$$

where $q \in (0,1)$, *then there exists one and only one element* $z \in X$ *such that* $z = Az = Sz = Tz$.

P r o o f. The proof is similar to the proof of Theorem 2 from |7|. Let $x_o \in X$. Since $AX \subseteq SX \cap TX$ we can define the sequence $\{x_n\}_{n\in N}$ from X in the following way:

$$Tx_{2k-1} = A^{n(x_{2k-2})}x_{2k-2}, \quad k \in N$$

$$Sx_{2k} = A^{n(x_{2k-1})}x_{2k-1}, \quad k \in N.$$

Further let $y_n = \begin{cases} Tx_{2k-1}, & n = 2k-1 \\ Sx_{2k}, & n = 2k \end{cases}$.

We shall prove that for every $a \in X$:

$$d(y_n, y_{n-1}, a) \leq q^{n-2} D(X)$$

for every $n \in N \setminus \{1\}$, where $D(X) = \sup_{x,y,z\in X} d(x,y,z)$. Let $n = 2k$. Then:

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$$d(y_n,y_{n-1},a) = d(Sx_{2k},Tx_{2k-1},a) =$$

$$= d(A^{n(x_{2k-1})}x_{2k-1},A^{n(x_{2k-2})}x_{2k-2},a) = d(A^{n(x_{2k-1})}T^{-1}Tx_{2k-1},$$

$$A^{n(x_{2k-2})}x_{2k-2},a) = d(A^{n(x_{2k-1})}T^{-1}A^{n(x_{2k-2})}x_{2k-2},A^{n(x_{2k-2})}x_{2k-1},a)$$

$$\leq qd(Sx_{2k-2},A^{n(x_{2k-1})}x_{2k-2},a) \leq \ldots \leq q^{2k-3}d(Sx_2,A^{n(x_{2k-1})}x_2,a)$$

$$\leq q^{2k-2}d(Tx_1,A^{n(x_{2k-1})}x_1,a) \leq q^{n-2}D(X) .$$

Similarly for $n = 2k-1$:

$$d(y_n,y_{n-1},a) = d(Tx_{2k-1},Sx_{2k-2},a) \leq qd(Tx_{2k-3},A^{n(x_{2k-2})}x_{2k-3})$$

$$\leq \ldots \leq q^{n-2}d(Tx_1,A^{n(x_{2k-2})}x_1,a) \leq q^{n-2}D(X) .$$

Now, we shall prove that $\{y_n\}_{n\ N}$ is a Cauchy sequence. Since:

$$d(y_n,y_{n+m},a) \leq d(y_n,y_{n+1},y_{n+m}) + d(y_n,y_{n+1},a) +$$

$$+ d(y_{n+1},y_{n+2},y_{n+m}) + d(y_{n+1},y_{n+2},a) + \ldots +$$

$$+ d(y_{n+m-2},y_{n+m-1},y_{n+m}) + d(y_{n+m-1},y_{n+m},a) \leq$$

$$\leq \sum_{k=n}^{n+m-2} q^{k-1}D(X) + \sum_{k=n}^{n+m-1} q^{k-1}D(X)$$

for every $n,m \in N$ and $q \in (0,1)$ it follows that $\{y_n\}_{n\in N}$ is a Cauchy sequence. So there exists $z \in X$ such that $\lim_{n\to\infty} y_n = z$ which means that for every $a \in X$: $\lim_{n\to\infty} d(y_n,z,a) = 0$.

Since $\{Sx_{2k}\}_{k\in N}$ and $\{Tx_{2k-1}\}_{k\in N}$ are subsequences of the sequence $\{y_n\}_{n\in N}$ it follows that for every $a \in X$:

(1) $$\lim_{k\to\infty} d(Sx_{2k},z,a) = \lim_{k\to\infty} (Tx_{2k-1},z,a) = 0.$$

Let us prove that $\lim_{k\to\infty} Ax_{2k} = \lim_{k\to\infty} A^2x_{2k} = z$. As in |7|:

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$$d(Sx_{2k},Ax_{2k},a) = d(A^{n(x_{2k-1})}x_{2k-1},AS^{-1}Sx_{2k},a) =$$

$$= d(A^{n(x_{2k-1})}x_{2k-1},AS^{-1}A^{n(x_{2k-1})}x_{2k-1},a) \leq qd(Tx_{2k-1},Ax_{2k-1},a)$$

$$= qd(A^{n(x_{2k-2})}x_{2k-2},AT^{-1}Tx_{2k-1},a) = qd(A^{n(x_{2k-2})}x_{2k-2},$$

$$AT^{-1}A^{n(x_{2k-2})}x_{2k-2},a) \leq q^2 d(Sx_{2k-2},Ax_{2k-2},a) \leq$$

$$\leq \ldots \leq q^{2k-1}d(Tx_1,Ax_1,a) \leq q^{2k-1}D(X)$$

and so:

$$d(Ax_{2k},z,a) \leq d(Ax_{2k},z,Sx_{2k}) + d(Ax_{2k},Sx_{2k},a) + d(Sx_{2k},z,a) \leq$$

$$\leq 2q^{2k-1}D(X) + d(Sx_{2k},z,a) .$$

Using (1) we conclude that $\lim_{k\to\infty} d(Ax_{2k},z,a) = 0$. Further:

$$d(Sx_{2k},A^2x_{2k},a) = d(A^{n(x_{2k-1})}x_{2k-1},A^2S^{-1}Sx_{2k},a) \leq$$

$$\leq qd(Tx_{2k-1},A^2x_{2k-1}) \leq \ldots \leq q^{2k-1}d(Tx_1,A^2x_1,a)$$

$$\leq q^{2k-1}D(X)$$

which implies that $\lim_{k\to\infty} d(A^2x_{2k},z,a) = 0$.

Then from:

$$d(Az,Sz,a) \leq d(Az,Sz,ASx_{2k}) + d(Az,ASx_{2k},a) + d(ASx_{2k},Sz,a) =$$

$$= d(Az,ASx_{2k},Sz) + d(Az,ASx_{2k},a) + d(SAx_{2k},Sz,a)$$

since A and S are continuous, it follows that:

$$d(Az,Sz,a) = 0, \quad \text{for every} \quad a \in X.$$

Using 1. we conclude that $Az = Sz$. Similarly we can prove that $Az = Tz$. From the continuity of A it is easy to obtain that $Az = z$. The uniqueness of the common fixed point z follows as in |7|.

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REMARK. From the proof of the Theorem it is easy to see that we can suppose instead of boundedness of the space X that for every $a \in X$: $\sup_{\substack{y \in O_A(a) \\ z \in X}} d(Ta,y,z) \leq M_a$, $O_A(a) = \{A^n a,\ n \in N \cup \{0\}\}$

The following theorem is a common fixed point theorem for $\{A_j\}_{j \in N}$, S and T.

THEOREM 2. *Let* (X,d) *be a complete 2-metric space*, $S,T : X \to X$ d *be continuous functional*, $A_j : X \to SX \cap TX$ $(j \in N)$, S *and* T *be continuous mapping,* A_j *commutes with* S *and* T $(j \in N)$ *and for every* i, j $\in N$ $(i \neq j)$:

$$d(A_i x, A_j y, a) \leq q\, d(Sx, Ty, a) \qquad \text{for every } x,y,a \in X$$

where $q \in [0,1)$. *Then there exists a unique common fixed point for the family* $\{A_j\}_{j \in N}$, S *and* T.

P r o o f. It is easy to see that there exists a sequence $\{x_n\}_{n \in N}$ in X such that:

$$Tx_{2n+1} = A_{2n+1} x_{2n}, \quad n \in N \cup \{0\}$$

$$Sx_{2n} = A_{2n} x_{2n-1}, \quad n \in N.$$

Let $y_n = \begin{cases} Tx_{2k+1}, & \text{for } n = 2k+1 \\ Sx_{2k}, & \text{for } n = 2k \end{cases}$. We shall prove that the sequence $\{y_n\}_{n \in N}$ is a Cauchy sequence. Since for $n = 2k$

$$d(y_{n-1}, y_n, a) = d(Tx_{2k-1}, Sx_{2k}, a) = d(A_{2k-1} x_{2k-2}, A_{2k} x_{2k-1}, a) \leq$$

$$\leq q\, d(Sx_{2k-2}, Tx_{2k-1}, a) = q\, d(A_{2k-2} x_{2k-3}, A_{2k-1} x_{2k-2}, a) \leq$$

$$\leq q^2 d(Tx_{2k-3}, Sx_{2k-2}, a) \leq \ldots \leq q^{n-2} d(Sx_2, Tx_1, a) = q^{n-2} d(y_2, y_1, a)$$

and similarly for $n = 2k+1$:

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$$d(y_{n-1},y_n,a) \leq q^{n-2}d(y_2,y_1,a) \text{ , for every } a \in X$$

it is easy to prove that $\{y_n\}_{n\in N}$ is a Cauchy sequence.

Namely, by standard arguments we have for every $m \in N$ that:

(2) $$d(y_1,y_2,y_m) = 0 .$$

This can be easily proved by induction. Further:

$$d(y_{m+1},y_m,y_n) \leq q^{m-1}d(y_2,y_1,y_n)$$

for every $m,n \in N$ and so from (2) it follows that:

$d(y_{m+1},y_m,y_n) = 0$, for every $m,n \in N$. It is well known that this implies $\lim_{m,n\to\infty} d(y_m,y_n,a) = 0$. From the completness of the space X it follows that there exists $z \in X$ such that:

$z = \lim_{n\to\infty} y_n$. Then $\lim_{n\to\infty} Tx_{2n-1} = \lim_{n\to\infty} Sx_{2n} = z$. Further:

$$d(Tz,TA_{2n}x_{2n-1},a) = d(Tz,A_{2n}Tx_{2n-1},a) = d(Tz,A_{2n}A_{2n-1}x_{2n-2},a)$$

tends to zero when $n \to \infty$ since T is continuous. So $Tz =$

$= \lim_{n\to\infty} A_{2n}A_{2n-1}x_{2n-1}$ and similarly $Sz = \lim_{n\to\infty} A_{2n+1}A_{2n}x_{2n-1}$.

Then from the inequality :

$$d(A_{2n}A_{2n-1}x_{2n-2},A_{2n+1}A_{2n}x_{2n-1},a) \leq qd(S(Sx_{2n}),T(Tx_{2n-1}),a)$$

and continuity of S,T and d it follows that:

$d(Tz,Sz,a) \leq qd(Sz,Tz,a)$, for every $a \in X$. This implies that $Tz = Sz$. Further from $(2m \neq n)$:

$$d(TSx_{2m},A_nz,a) = d(A_{2m}(Tx_{2m-1}),A_nz,a) \leq qd(S(Tx_{2m-1}),Tz,a)$$

when $m \to \infty$ we conclude that $Tz = A_nz = Sz$, for every $n \in N$. Let us prove that $A_nz = A_nA_nz$, for every $n \in N$. We have:

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$$d(A_n z, A_n A_n z, a) \leq d(A_n z, A_n A_n z, A_{2m} Tx_{2m-1}) +$$

$$+ d(A_n z, A_{2m} Tx_{2m-1}, a) + d(A_{2m} Tx_{2m-1}, A_n A_n z, a) \leq$$

$$\leq d(A_n z, A_n A_n z, T(A_{2m} x_{2m-1})) + d(A_n z, T(A_{2m} x_{2m-1}), a) +$$

$$+ qd(S(Tx_{2m-1}), TA_n z, a)$$

and when $m \to \infty$ we obtain:

$$d(A_n z, A_n A_n z, a) \leq d(A_n z, A_n A_n z, Tz) + d(A_n z, Tz, a) +$$

$$+ qd(Sz, TA_n z, a) = qd(A_n z, A_n A_n z, a) .$$

From this we conclude that $A_n z = A_n A_n z$, for every $n \in N$ and so $A_n z = Tz = Sz$ is the common fixed point for the family $\{A_n\}_{n \in N}$ S and T.

Let $w \in X$ be such that $A_n w = Tw = Sw$ and $u = Tz = Sz = A_n z$. Then for $i \neq j$ we have:

$$d(w,u,a) = d(A_i w, A_j u, a) \leq q\, d(Sw, Tu, a) = d(w,u,a) .$$

This implies that $u = w$.

Now, we shall prove a fixed point theorem in 2-Banach spaces. Let L be a linear space of dimension greater than one and $||.,.||$ a real function on $L \times L$ which satisfies the following conditions:

1. $||a,b|| = 0$ if and only if a and b are linearly dependent.
2. $||a,b|| = ||b,a||$, for every a,b in L.
3. $||ta,b|| = |t|\,||a,b||$, for every a,b in L and real number t.
4. $||a+b,c|| \leq ||a,c|| + ||b,c||$, for every a,b,c in L.

Then $||\ ,\ ||$ is called a 2-norm on L and $(L, ||\ ,\ ||)$ is called a 2-normed space.

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Every 2-normed space is a 2-metric space with 2-metric d defined by $d(a,b,c) = ||b-a,c-a||$.

In |3| a natural topology in 2-metric space and so in 2-normed space is introduced. For a 2-normed space $(L,||,||)$ this is a locally convex T_2- topology in which the fundamental system of neighbourhoods of zero in L is given by the family:

$$W_\Sigma(0) = \{a, a \in L,\ ||a,a_j|| < \varepsilon_j,\ \text{for every } j \in \{1,2,\ldots,m\}\}$$

where

$$\Sigma = \{(a_1,\varepsilon_1),\ldots,(a_m,\varepsilon_m)\}(a_j \in L,\ j \in \{1,2,\ldots,m\},\ \varepsilon_j > 0,$$
$$j \in \{1,2,\ldots,m\}) .$$

A subset M of L is bounded if and only if every semi-norm $(b \in L)$ μ_b defined by $\mu_b(a) = ||a,b||$ $(a \in L)$ is bounded on M. Every 2-normed space $(L,||,||)$ is uniformisable since it is a topological vector space. For the base of this uniformity we take the family:

$$\{(a,b) \mid b-a \in W_\Sigma(0)\}\ ,\quad \Sigma = \{(a_1,\varepsilon_1),\ldots,(a_m,\varepsilon_m)\}$$
$$(a_j \in L, \varepsilon_j > 0 \text{ for every } j \in \{1,2,\ldots,m\}).$$

So a Moore-Smith sequence $\{b_i\}_{i \in I}$ from L is Cauchy sequence if and only if for every $b \in L$ and every $\varepsilon > 0$ there exists a $j \in I$ so that $||b_i - b_{i'},b|| < \varepsilon$ for every $i, i' \in I$ so that $i, i' \geq j$.

A 2-normed space $(L,||,||)$ is complete if every Cauchy sequence is convergent and such 2-normed space is called a 2-Banach space.

From the uniforme structure point of view the class of Banach space of dimension greater than 1 coincide with the class of strong locally bounded 2-Banach spaces |4| .

THEOREM 3. *Let* $(X,||,||)$ *be 2-Banach space,* S *and* T *linear continuous mappings from* X *into* X, *for every* $j \in N$, $A_j : X \to SX \cap TX$ *continuous mapping,* A_j *commutes with* S *and* T *and for every* $j \in N$, A_jX *be bounded. Further, assume that the following conditions are satisfied:*

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1. *For every* $i,j \in N$ $(i \neq j)$, *every* $x,y \in X$ *and every* $a \in X$:

$$\| A_i x - A_j y, a \| \leq \| Sx - Ty, a \|$$

2. *There exist* $j_o \in N$, $m_o \in N$ *and* $k_o \in N$ *so that the set* $A_{j_o}^{m_o} M$ *is relatively sequentially compact where:*

$$M = \overline{[0,1]\, A_{k_o} X}.$$

Then there exists $x \in X$ *so that*

$$x = A_j x = Sx = Tx \ , \ \textit{for every } j \in N \ .$$

P r o o f. Let $\{r_n\}_{n \in N}$ be a sequence of real numbers from $(0,1)$ so that $\lim_{n \to \infty} r_n = 1$ and $A_{j,n} x = r_n A_j x$, $x \in X$, $(j,n) \in N \times N$. Then:

$$\| A_{j,n} x - A_{k,n} y, a \| = r_n \| A_j x - A_k y, a \| \leq r_n \| Sx - Ty, a \|$$

for $j \neq k$, every $n \in N$ and every $(x,y,a) \in X \times X \times X$. It is obvious that $A_{j,n}$ is commutative with S and T and so for every $n \in N$ there exists $x_n \in X$ so that:

$$x_n = A_{j,n} x_n = Sx_n = Tx_n, \text{ for every } n \in N \text{ every } j \in N.$$

Let us prove that $\lim_{n \to \infty} \| x_n - A_j x_n, a \| = 0$, for every $a \in X, j \in N$. Since $A_j X$ is bounded for every $j \in N$ there exists $M(j,a)$ so that $\| A_j x_n, a \| \leq M(j,a)$, for every $n \in N$. So we have that:

$$\| x_n - A_j x_n, a \| = \| r_n A_j x_n - A_j x_n, a \| =$$

$$= \| (r_n - 1) A_j x_n, a \| \leq |r_n - 1| \, M(j,a)$$

and since $\lim_{n \to \infty} r_n = 1$ it follows that $\lim_{n \to \infty} \| x_n - A_j x_n, a \| = 0$.

Further, let us prove that for every $(j,i,n,k) \in N \times N \times N \times N$ $(j \neq i)$:

$$(3) \quad \| A_j^{k+1} x_n - A_i^k x_n, a \| \leq \| A_j x_n - x_n, a \| \text{ for every } a \in X.$$

The proof will be given by induction. For $k = 1$ we have:

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$$\| A_j^2 x_n - A_i x_n, a\| \leq \| S(A_j x_n) - Tx_n, a\| = \| A_j(Sx_n) - Tx_n, a\| =$$
$$= \| A_j x_n - x_n, a\| .$$

Suppose that $\| A_j^k x_n - A_i^{k-1} x_n, a\| \leq \| Ax_n - x_n, a\|$ for every $(i,j,n) \in N \times N \times N$ $(i \neq j)$. Then (3) follows from:

$$\| A_j^{k+1} x_n - A_i^k x_n, a\| \leq \| S(A_j^k x_n) - T(A_i^{k-1} x_n), a\| =$$
$$= \| A_j^k Sx_n - A_i^{k-1} Tx_n, a\| = \| A_j^k x_n - A_i^{k-1} x_n, a\| \leq \| A_j x_n - x_n, a\|$$

and so:

$$(4) \qquad \| A_{j_o}^{m_o} x_n - x_n, a\| \leq \sum_{s=j_o}^{j_o+m_o} \| A_s x_n - x_n, a\| .$$

From (4) it follows that $\lim_{n\to\infty} \| A_{j_o}^{m_o} x_n - x_n, a\| = 0$. Further, for every $n \in N$, $x_n = r_n A_{k_o} x_n$ and so $x_n \in M$ for every $n \in N$. Since the set $A_{j_o}^{m_o} M$ is sequentially compact it follows that there exists $x \in X$ so that for some subsequence $\{x_{n_k}\}_{k\in N}$ of $\{x_n\}_{n\in N}$ we have:

$$(5) \qquad \lim_{k\to\infty} \| A_{j_o}^{m_o} x_{n_k} - x, a\| = 0.$$

The relation (5) implies that $\lim_{k\to\infty} \| x_{n_k} - x, a\| = 0$. From this it is easy to prove that x is such that:

$$x = A_j x = Sx = Tx, \quad \text{for every} \quad j \in N.$$

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REZIME

TEOREME O ZAJEDNIČKOJ NEPOKRETNOJ TAČKI U 2-METRIČKIM PROSTORIMA

U ovom radu je dokazana teorema o zajedničkoj nepokretnoj tački preslikavanja A,S i T u 2-metričkim prostorima.

TEOREMA 1. *Neka je* (X,d) *ograničen, kompletan 2-metrički prostor,* S *i* T *obostrano jednoznačna i neprekidna preslikavanja* X *u* X , $A : X \to SX \cap TX$ *neprekidno preslikavanje a* T *i* S *komutativna sa* A . *Ako za svako* $x \in X$ *postoji* $n(x) \in N$ *tako da je za svako* $y \in X$ *i svako* $a \in X$:

$$d(A^{n(x)}x, A^{n(x)}y, a) \leq q \min\{d(Sx,Ty,a), d(Tx,Sy,a)\}$$

gde je $q \in (0,1)$ *tada postoji jedan i samo jedan elemenat* $z \in X$ *takav da je* $z = Az = Sz = Tz$.

Takodje je dokazana sledeća teorema:

TEOREMA 2. *Neka je* (X,d) *kompletan 2-metrički prostor sa neprekidnom funkcionelom* d, $A_j : X \to SX \cap TX$ $(j \in N)$ *gde su* $S,T : X \to X$ *neprekidna preslikavanja komutativna sa* A *i za svako* $i,j \in N (i \neq j)$

$$d(A_i x, A_j y, a) \leq q d(Sx,Ty,a) \qquad \text{za svako} \quad x,y,a \in X$$

gde je $q \in (0,1)$. *Tada postoji jedinstvena zajednička nepokretna tačka familije* $\{A_j\}_{j \in N}$, S i T.

Primenom Teoreme 2 dokazana je sledeća teorema.

TEOREMA 3. *Neka je* $(X, \|\,,\,\|)$ *2-Banachov prostor,* S *i* T *linearna neprekidna preslikavanja* X *u* X, *za svako* $j \in N$ *je* $A_j : X \to SX \cap TX$ *neprekidno preslikavanje,* A_j *je komutativno preslikavanje sa* S *i* T *i za svako* $j \in N$ *je* $A_j X$ *ograničen skup. Pretpostavimo, dalje, da su zadovoljeni sledeći uslovi:*

1. *Za svako* $i,j \in N (i \neq j)$, *za svako* $x,y \in X$ *i za svako* $a \in X$

$$\|A_i x - A_j y, a\| \leq \|Sx - Ty, a\|$$

2. *Postoje* $j_o \in N$, $m_o \in N$ *i* $k_o \in N$ *tako da je* $A_{j_o}^{m_o} M$ *relativno kompaktan gde je* $M = [0,1] A_{k_o} X$.

Tada postoji $x \in X$ *tako da je:* $x = A_j x = Sx = Tx$ *za svako* $j \in N$.

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A LERAY-SCHAUDER PRINCIPLE FOR MULTIVALUED MAPPINGS IN TOPOLOGICAL VECTOR SPACES

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ABSTRACT

Using a result of Schöneberg |15| and a fixed point theorem from |2| we prove in this paper a fixed point theorem for multivalued mappings in not necessarily locally convex topological vector spaces.

INTRODUCTION

In recent time there is an increasing interest in the fixed point theory in not necessarily locally convex topological vector spaces and some results for singlevalued and multivalued mappings are obtained in |1|, |2|, |3|, |4|, |5|, |6|, |7|, |8|, |10| and |15| . It is proved in |8| that for admissible (see |10|) topological vector spaces many results from the fixed point theory in locally convex topological vector spaces can be generalized. Since there are many important nonlocally convex topological vector spaces such as $L^p(0< p < 1)$, $S(0,1)$- the space of measurable functions, Hardy's class, SL_ϕ, KL_ϕ |12|, it is of interest to obtain fixed point theorems for not necessarily locally convex topological vector spaces.

Using a result of Schöneberg |15| and a fixed point theorem from |2|, we shall prove in this paper a fixed point theorem for multivalued mappings in not necessarily locally convex topological vector spaces which are Hausdorff.

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A LERAY-SCHAUDER PRINCIPLE

First, we shall give some known results, notations and definitions.

In |2| we have proved the following fixed point theorem for multivalued mappings in not necessarily locally convex topological vector spaces.

THEOREM A. *Let* E *be a Hausdorff topological vector space,* $\mathcal{U}$ *be the fundamental system of neighborhoods of zero in* E, K *be a closed and convex subset of* E, $R(K)$ *be the family of all nonempty, closed and convex subset in* K, $F : K \to R(K)$ *be an upper semi-continuous mapping,* $\overline{F(K)}$ *be compact and the following condition be satisfied: (*) For every* $V \in \mathcal{U}$ *there is* $U \in \mathcal{U}$ *so that the convex hull of* $U \cap (F(K)-F(K))$ *is in* V. *Then there exists* $x \in K$ *so that* $x \in Fx$.

In |2| some applications of the above fixed point theorem are given. If the space E is a locally convex topological vector space the condition (*) from the above theorem is satisfied for every $F : K \to 2^K$ since we can take for $U \in \mathcal{U}$ the neighborhood V, if we suppose that co U = U (co-convex hull) for every $U \in \mathcal{U}$.

Now, let us introduce the following definition.

DEFINITION 1. *Let* E *be a topological vector space,* $\mathcal{U}$ *be the fundamental system of neighborhoods of* $0 \in E$, $K \subseteq Z.K$ *is said to be of Zima's type if and only if for every* $V \in \mathcal{U}$ *there exists* $U \in \mathcal{U}$ *such that co* $(U \cap (K-K)) \subseteq V$.

If E is a locally convex topological vector space and $K \subseteq E$ then K is of Zima's type. K.Zima has given in |16| a subset K of a paranormed space E which is in fact of Zima's type in the sense of Definition 1. Now, we shall give an example of a subset of Zima's type in a nonlocally convex topological vector space.

Let E be a vector space over the real or complex number field and $\| \|^* : E \to [0, \infty)$ so that the following conditions are satisfied:

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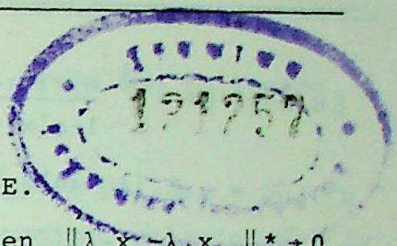

1. $\|x\|^* = 0 \Leftrightarrow x = 0$.

2. $\|-x\|^* = \|x\|^*$, $x \in E$.

3. $\|x+y\|^* \leq \|x\|^* + \|y\|^*$, $x,y \in E$.

4. If $\|x_n - x\|^* \to 0$ and $\lambda_n \to \lambda_o$ then $\|\lambda_n x_n - \lambda_o x_o\|^* \to 0$.

Then $(E, \|\ \|^*)$ is a paranormed space which is also a topological vector space in which the fundamental system V of neighborhoods of zero in E is given by:

$$V = \{V_\varepsilon\}\varepsilon > 0, \quad V_\varepsilon = \{x \mid \|x\|^* < \varepsilon,\ x \in E\}.$$

If $K \subseteq E$, where $(E, \|\ \|^*)$ is a paranormed space, and there exists $C(K) > 0$ such that |16|:

$\|tx\|^* \leq C(K)t\|x\|^*$, for every $x \in K-K$ and every $t \in [0,1]$, in |2| it is proved that K is of Zima's type.

Now, let $(E, \|\ \|^*)$ be the space $(S(0,1), \|\ \|^*)$ $(S(0,1)$ the space of equivalence classes of finite measurable functions on $[0,1]$ with measure μ) and for every $\hat{x} \in S(0,1)$:

$$\|\hat{x}\|^* = \int_0^1 \frac{|x(t)|}{1+|x(t)|}\,\mu(dt), \quad \{x(t)\} \in \hat{x}\ .$$

It is known that S(0,1) is an admissible topological vector space |13| and the convergence in the paranorme is the convergence in the measure.

Let $\alpha > 0$ and $K_\alpha = \{\hat{x} \mid \hat{x} \in S(0,1),\ |x(t)| \leq \alpha,\ t \in [0,1]\}$. We shall show that K_α is of Zima's type and that $C(K_\alpha) = 1 + 2\alpha$ which means that:

$\|s(\hat{x}-\hat{y})\|^* \leq (1+2\alpha)s\|\hat{x}-\hat{y}\|^*$, for every $0 \leq s \leq 1$ and every $\hat{x},\hat{y} \in K_\alpha$. Let $\{x(t)\} \in \hat{x}$ and $\{y(t)\} \in \hat{y}$ so that:

$$|x(t)| \leq \alpha, \quad |y(t)| \leq \alpha, \quad t \in [0,1]\ .$$

Then $1 + |x(t)-y(t)| \leq (1+2\alpha) + (1+2\alpha)s|x(t)-y(t)| =$

$= (1+2\alpha)(1+s|x(t)-y(t)|)$

and so:

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$$\|s(\hat{x}-\hat{y})\|^* = \int_0^1 \frac{s|x(t)-y(t)|}{1+s|x(t)-y(t)|}\,\mu(dt) \le (1+2\alpha)s \int_0^1 \frac{|x(t)-y(t)|}{1+|x(t)-y(t)|}\,\mu(dt)$$

$$= (1+2\alpha)s\,\|\hat{x}-\hat{y}\|^* .$$

Let E be a topological vector space, $K \subseteq E$ and 2^K be the family of all nomepty subsets of K. If $U \subseteq K$ then $cl_K(U)$ is the closure (in K) of U and $\partial_K U$ denotes the boundary of U in K. If C is a lattice with a minimal element θ then a map $\chi : 2^K \to C$ is called a measure of noncompactness of K provided that the following conditions are satisfied for any $X, Y \in 2^K$:

1. $\chi(X) = \theta$ if and only if X is relatively compact.
2. $\chi(\overline{co}(X)) = \chi(X)$, where $\overline{co}(X)$ denotes the convex closure of X.
3. $\chi(X \cup Y) = \max\{\chi(X), \chi(Y)\}$.

If χ is a measure of noncompactness of K, $D \subseteq K$ and $g : D \to 2^K$ then g is called χ-condensing if and only if the following implication holds:

$$X \subseteq D,\ \chi(X) \le \chi(g(X)) \Rightarrow X \text{ is relatively compact.}$$

In |15| the following theorem is proved.

THEOREM B. *Let χ be a measure of noncompactness of K, M be a set of nonempty compact subsets of K, $U \subseteq K$ be an open neighborhood of zero in E, $0 \in K$ and $H : [0,1] \times cl_K(U) \to M$ be an upper semicontinuous mapping. Suppose that the following conditions are satisfied:*

(i) $x \notin H(t,x)$ *for* $t \in [0,1]$ *and* $x \in \partial_K U$.

(ii) $X \subset cl_K(U)$ *and* $\chi(X) \le \chi(H([0,1] \times X))$ *imply that* X *is relatively compact.*

(iii) For every $t > 1$ *and* $x \in \partial_K U$, $tx \notin H(1,x)$.

(iv) $tM \in M$ *for* $t \in [0,1]$ *and* $M \in M$.

Then there exists an upper semicontinuous mapping $g : K \to M$ *such that for* $x \in K$:

$x \in g(x)$ *if and only if* $x \in cl_K(U)$ *and* $x \in H(0,x)$.

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Using Theorems A and B we shall prove the following fixed point theorem.

THEOREM *Let* $K, U \in \mathcal{U}$ *and* χ *be as in Theorem B and* $\mathcal{M}'$ *the family of all nonempty, convex and compact subsets of* K *so that* $\operatorname{co}(\{0\} \cup H([0,1] \times \operatorname{cl}_K(U)))$ *is of Zima's type where* H *is an upper semicontinuous mapping from* $[0,1] \times \operatorname{cl}_K(U)$ *into* $\mathcal{M}'$ *so that (i), (ii) and (iii) in Theorem* B *are satisfied. Then there exists* $x \in \operatorname{cl}_K(U)$ *such that* $x \in H(0,x)$.

P r o o f. The family $\mathcal{M}'$ has the property (iv) from Theorem B. Namely $0 \in K$ and K is convex, so for every $M \subseteq K$ it follows that $tM \subseteq K$ for every $t \in [0,1]$ and so $M \in \mathcal{M}'$ implies that $tM \in \mathcal{M}'$. From Theorem B it follows that there exists an upper semicontinuous χ-condensing mapping $g: K \to \mathcal{M}'$ so that for all $x \in K$:

$$(1) \qquad x \in g(x) \iff x \in \operatorname{cl}_K(U), \quad x \in H(0,x).$$

Now, we shall prove that the set of fixed points of the mapping g, $\operatorname{Fix}(g)$, is nonempty. Let us prove that there exists a closed and convex subset $T \subseteq K$ so that $g(T) \subseteq T$ and $\overline{g(T)}$ is compact. In order to do this we shall use Remark 6 from Hahn's paper |6|. The measure χ has the following properties:

(a) $X \subseteq Y (X, Y \subseteq K) \Rightarrow \chi(X) \leq \chi(Y)$.

(b) $\chi(X \cup \{y_0\}) = \chi(X)$, for every $X \subseteq K$, $y_0 \in K$.

Indeed $X \subseteq Y$ implies $\chi(X \cup Y) = \chi(Y) = \max\{\chi(X), \chi(Y)\}$ and so $\chi(Y) \geq \chi(X)$. Further $\chi(X \cup \{y_0\}) = \max\{\chi(X), \chi(\{y_0\}) = \chi(X)$ since $\chi(\{y_0\}) = 0$. If $M = K$ in Remark 6 |6| as in |6| it follows that there exists a nonempty, closed and convex subset T_0 of E such that $K \cap T_0 \neq \emptyset$, $g(K \cap T_0) \subseteq K \cap T_0$ and $\chi(K \cap T_0) \leq \chi(g(K \cap T_0))$. Since the mapping g is χ-condensing it follows that $K \cap T_0$ is compact and so from $\overline{g(K \cap T_0)} \subseteq K \cap T_0$, it follows that $\overline{g(K \cap T_0)}$ is compact. Further in the proof of Theorem B the mapping g is defined in the following way ($\lambda: K \to [0,1]$ is defined in |15|)

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$$g(x) = \begin{cases} H(2\lambda(x),x) & \lambda(x) \leq \frac{1}{2}\ ,\ x \in cl_K(U) \\ 2(1-\lambda(x))H(1,x) & \lambda(x) \geq \frac{1}{2}\ ,\ x \in cl_K(U) \\ 0 & x \notin cl_K(U) \end{cases}$$

From this it follows that $g(K) \subseteq co(\{0\} \cup H([0,1] \times cl_K(U)))$. Since the set $co(\{0\} \cup H([0,1] \times cl_K(U)))$ is of Zima's type we conclude that the set $g(K)$ is of Zima's type. So, on the set $T = K \cap T_o$ the mapping g satisfies all the conditions of Theorem A and there exists $x \in T$ so that $x \in g(x)$. From (1) it follows that there exists $x \in cl_K(U)$ such that $x \in H(0,x)$.

From the Theorem we obtain the following Corollary.

COROLLARY 1. *Let* E *be a complete topological vector space,* $0 \in K \subseteq E$ *and* K *be a closed and convex of Zima type subset,* U *be an open neighborhood of zero in* E *and* $U \subset K$, $\mathcal{M}'$ *be as in the Theorem and* $H : [0,1] \times cl_K(U) \to \mathcal{M}'$ *so that the following conditions are satisfied:*

(A) *For every* $t \in [0,1]$ *and every* $x \in \partial_K U$, $x \notin H(t,x)$.

(B) $\overline{H([0,1] \times cl_K(U))}$ *is compact.*

(C) *For every* $t > 1$ *and every* $x \in \partial_K U$, $tx \notin H(1,x)$.

Then there exists $x \in cl_K(U)$ *such that* $x \in H(0,x)$.

P r o o f. Let $C = \{0,1\}$ and $\chi : 2^K \to C$ be defined in the following way:

$$\chi(X) = \begin{cases} 0 & \bar{X} \text{ is compact} \\ 1 & \bar{X} \text{ is not compact.} \end{cases}$$

Then χ is a measure of noncompactness. Indeed 1. follows from the definition of the mapping χ. Let us prove 2.. Since $A \subset B$ implies $\chi(A) \leq \chi(B)$ we have that $\chi(X) \leq \chi(\overline{co}\ X)$ for every $X \in 2^K$. Further, if $\bar{X}$ is compact it follows that $\overline{co}\ X$ is compact as we have proved in [4]. So we have that $\chi(X) = 1$ implies that $\chi(\overline{co}\ X) = 1 = \chi(X)$. If X is not relatively compact then $\chi(\overline{co}X) < 1 = \chi(X)$ and so 2. is proved. 3. is obvious. Further, we have the following implication:

$$X \subseteq cl_K(U),\ \chi(X) \leq \chi(H([0,1] \times X)) \Rightarrow \bar{X} \text{ is compact.}$$

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Indeed, from $H([0,1] \times X) \subseteq H([0,1] \times cl_K(U))$ and from the compactness of the set $H([0,1] \times cl_K(U))$ it follows that $\chi(H([0,1] \times X) = 0$, for every $X \subseteq cl_K(U)$. From this we conclude that the inequality $\chi(X) \leq \chi(H([0,1] \times X)$ implies $\chi(X) = 0$ and so $\bar{X}$ is compact. From the Theorem it follows that there exists $x \in cl_K(U)$ so that $x \in H(0,x)$.

From Corollary 1 we obtain the following Corollary.

COROLLARY 2. *Let* $(E, \|\ \|^*)$ *be a complete paranormed space,* $0 \in K \subseteq E$, K *be a closed and convex subset of* E *such that there exists* $C(K) > 0$ *so that:* $\|tx\|^* \leq t\, C(K)\, \|x\|^*$, *for every* $t \in [0,1]$ *and every* $x \in K-K$, *and* U, H *and* χ, M' *be as in Corollary 1. Then there exists* $x \in cl_K(U)$ *such that* $x \in H(0,x)$.

Now, we shall give another corollary of the Theorem using some results from the theory of topological semifields. First, we shall recall some definitions from |9|.

Let $\mathbb{R}$ be the set of all real numbers, E be a vector space over $\mathbb{K}$ (real or complex number field), $\mathbb{R}_\Delta$ be the set of all mappings from Δ into $\mathbb{R}$ with the Tychonoff product topology and the operations addition and scalar multiplication as usual. If $f, g \in \mathbb{R}_\Delta$, we shall say that $f < g$ if and only if $f(t) \leq g(t)$ for $t \in \Delta$ and $f \neq g$. By $\mathbb{P}_\Delta$ we shall denote the cone of non-negative elements in $\mathbb{R}_\Delta$. In |9| the notion of a paranormed space over a topological semifield is introduced. For the semifield $\mathbb{R}_\Delta$ we obtain the following definition.

DEFINITION 2. *The triplet* $(E, \|\ \|, \Phi)$ *is a paranormed space over a topological semifield* $\mathbb{R}_\Delta$ *if and only if* $\|\ \| : E \to \mathbb{P}_\Delta$ *and* Φ *is a linear, continuous, positive mapping from* $\mathbb{R}_\Delta$ *into* $\mathbb{R}_\Delta$ *such that the following conditions are satisfied:*

1. $\|x\| = 0 \iff x = 0$.
2. $\|\lambda x\| = |\lambda|\, \|x\|$, *for every* $x \in E$ *and every* $\lambda \in \mathbb{K}$
3. $\|x+y\| \leq \Phi(\|x\|) + \Phi(\|y\|)$, *for every* $x, y \in E$.

Let us denote by $\mathcal{U}$ the family of neighborhoods of zero in $\mathbb{R}_\Delta$ and for every $U \in \mathcal{U}$ we shall denote the set

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$$\{x \mid x \in E, \ \|x\| \in U\}$$

by V_U. Then E is a topological vector space in which $V = \{V_U\}_{U \in \mathcal{U}}$ is the family of neighborhoods of zero in E. In |9| it is proved that every Hausdorff topological vector space E is a paranormed space $(E, \|\ \|, \Phi)$ over a topological semified $\mathbb{R}_\Delta$ (for some set Δ) and we shall say that the triplet $(E, \|\ \|, \Phi)$ is the associated paranormed space. In |1| we have introduced the following definition.

DEFINITION 3. *Let* E *be a Hausdorff topological vector space and* $(E, \|\ \|, \Phi)$ *be the associated paranormed space. The set* $K \subseteq E$ *is of* Φ*-type if and only if for every* $n \in \mathbb{N}$*:*

$$\| \sum_{i=1}^{n} t_i x_i \| \leq \sum_{i=1}^{n} t_i \, \Phi(\|x_i\|)$$

for every $x_i \in K-K$ $(i=1,2,\ldots,n)$ *and every* $t_i \in [0,1]$ $\sum_{i=1}^{n} t_i = 1$.

In |1| it is proved that every subset of Φ is admissible in the sense of V.Klee. A fixed point theorem for such sets is also proved in |1|.

LEMMA *Let* E *be a Hausdorff topological vector space and* K *be a subset of* E *of* Φ *type. Then* K *is of Zima's type.*

P r o o f. We shall prove that for every $W \in V$ there exists $U \in V$ so that $co(U \cap (K-K)) \subset W$. Let $W \in V$ and $(E, \|\ \|, \Phi)$ be the associated paranormed space over the topological semifield $\mathbb{R}_\Delta$. For every $\varepsilon > 0$ and every finite subset Δ' of Δ, let us denote by $U_{\Delta', \varepsilon}$ the set $\{\tilde{x} \mid \tilde{x} \in \mathbb{R}_\Delta, \ \tilde{x}(t) < \varepsilon, \text{ for every } t \in \Delta'\}$. From the definition of the family V it follows that there exists $\{t_1, t_2, \ldots, t_n\} = \Delta' \subset \Delta$ and $\varepsilon > 0$ so that:

$$\|u\| \in U_{\Delta', \varepsilon} \Rightarrow u \in W \quad .$$

The mapping Φ is linear and continuous and so let U_1 be such neighborhood of zero in $\mathbb{R}_\Delta$ that:

$$\|u\| \in U_1 \Rightarrow \Phi(\|u\|) \in U_{\Delta', \varepsilon} \ .$$

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If U_2 is a symmetric neighborhood of zero in E such that $U_2 \subset V_{U_1}$ then:

$$co(U_2 \cap (K-K)) \subset W .$$

Let $u \in co(U_2 \cap (K-K))$. Then $u = \sum_{i=1}^{n} s_i u_i$, where $u_i \in U_2 \cap (K-K)$ for every $i \in \{1,2,\ldots,n\}$ and $s_i \in [0,1]$ $(i \in \{1,2,\ldots,n\})$ so that $\sum_{i=1}^{n} s_i = 1$. Since the set K is of Φ type it follows that:

$$\|u\|(t) = \| \sum_{i=1}^{n} s_i u_i \|(t) \leq \sum_{i=1}^{n} s_i \phi(\|u_i\|)(t) < \varepsilon$$

for every $t \in \Delta'$.

COROLLARY 3. *Let* E *be a complete topological vector space. Further, let* $0 \in K \subseteq E$, K *be a closed and convex subset of* Φ *type and* U , H *and* χ, M' *be as in Corollary 1. Then there exists* $x \in cl_K(U)$ *such that* $x \in H(0,x)$.

P r o o f. From the Lemma it follows that all the conditions of Corollary 1 are satisfied.

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REZIME

LERAY-SCHAUDEROV PRINCIP ZA VIŠEZNAČNA PRESLIKAVANJA U VEKTORSKO TOPOLOŠKIM PROSTORIMA

Korišćenjem rezultata Schöneberga |15| i teoreme o nepokretnoj tački iz |2| u ovom radu je dokazana sledeća teorema.

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TEOREMA. *Neka je* E *vektorsko topološki prostor,* $K \subseteq E$, χ *mera nekompaktnosti nad* K, $0 \in K$, $U \subseteq K$ *i* U *je okolina nule u* E, $\mathcal{M}'$ *familija nepraznih, konveksnih i kompaktnih podskupova od* K, $H : [0,1] \times cl_K(U) \to \mathcal{M}'$, $co(\{0\} \cup H([0,1] \times cl_K(U)))$ *zadovoljava Zimin uslov (Definicija 1) tako da su zadovoljeni sledeći uslovi:*

(i) $x \notin H(t,x)$ *za* $t \in [0,1]$ *i* $x \in \partial_K U$.

(ii) $X \subset cl_K(U)$ *i* $\chi(X) \leq \chi(H([0,1] \times X))$ *implicira da je skup* X *relativno kompaktan.*

(iii) *Za svako* $t > 1$ *i* $x \in \partial_K U$, $tx \notin H(1,x)$.

Ako je preslikavanje H *od gore poluneprekidno tada postoji* $x \in cl_K(U)$ *tako da je* $x \in H(0,x)$.

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ON THE EQUATION $Tx = F(x,Q(Tx))$ IN LOCALLY CONVEX SPACES

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ABSTRACT

In |6| two theorems on the existence of a solution of equations $Tx = F(x,Tx)$ and $Tx = F(x,Q(Tx))$ in locally convex spaces are proved. Using fixed point theorems from |4| and |7| we shall prove in this paper, some generalizations of Rzepecki's results from |6|.

In |1|, |2|, |7| and |3|, |4| some generalizations of the Banach contraction principle in locally convex spaces are proved. Some applications of such a result in the theory of Mikusinski operators are given in |3|.

Here we shall use the following fixed point theorem proved in |7|.

THEOREM A. *Let $(X,\{d_i\}_{i\in I})$ be a sequentially complete Hausdorff uniformizable space, T be continuous mapping X into X satisfying the following conditions:*

1) *For each $i \in I$ there exist $f(i) \in I$ and a nondecreasing function $q_i : \mathbb{R}^+ = [0,\infty) \to [0,1]$ so that*

$$d_i(Tx,Ty) \le q_i(d_{f(i)}(x,y))\, d_{f(i)}(x,y)$$

for every $x,y \in X$.

2) *For each $i \in I$ and $t \in \mathbb{R}^+$, $\overline{\lim_{n\to\infty}} q_{f^n(i)}(t) < 1$.*

3) *There is an element $x_o \in X$ such that:*

$$\sup_{n\in N} d_{f^n(i)}(x_o,Tx_o) = K_i \in \mathbb{R} \text{ , for every } i \in I.$$

Then there exists a unique $x^* \in X$ such that $x^* = Tx^*$ and for each $i \in I$:

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$$\sup_{n\in N} d_{f^n(i)}(x_o, x^*) = S_i \in \mathbb{R} .$$

In the following Lemma A is an index set and $\{f_a\}_{a\in A}$ is a net of transformations.

LEMMA 1. *Let* E *be a Hausdorff locally convex topological vector space in which the topology is defined by the family* $\{p_i\}_{i\in I}$ *of seminorms and* $g : I \to I$. *Further let* M *be a closed sequentially complete subset of* E, *for every* $a \in A$, $f_a : M \to M$ *so that* $\lim_{a\in A} f_a x$ *exists for every* $x \in M$ *and the following conditions are satisfied:*

a) *For every* $i \in I$ *there exists* $q_i : \mathbb{R}^+ \to [0,1]$ *which is a nondecreasing function for which is* $\overline{\lim}_{n\to\infty} q_{g^n(i)}(t) < 1$ *for every* $i \in I$ *and every* $t \in \mathbb{R}^+$ *and for every* $x,y \in M$:

(1) $p_i(f_a x - f_a y) \le q_i(p_{g(i)}(x-y)) p_{g(i)}(x-y)$ *for every* $a \in A$.

b) *For every* $i \in I$ *there exists a continuous function* $h_i : E \to \mathbb{R}^+$ *such that* h_i *maps bounded set into bounded set* $h_i(0) = 0$ *and* $p_{g^n(i)}(x) \le h_i(x)$, *for every* $x \in E$ *and every* $n \in N \cup \{0\}$, *and for every* $x \in M$ *is* $\{f_a x \mid a \in A\}$ *bounded.*
If $f_o x = \lim_{a\in A} f_a x$, $x \in M$, $f_a x_a = x_a$ *for every* $a \in A$, $x_o = f_o x_o$ *then* $\lim_{a\in A} x_a = x_o$.

P r o o f. From (1) it follows that:

$$p_i(f_o x - f_o y) \le q_i(p_{g(i)}(x-y)) p_{g(i)}(x-y)$$

for every $x,y \in M$ and every $i \in I$. Applying Theorem A, from b) we conclude that for every $b \in A \cup \{0\}$ there exists one and only one $x_b \in M$ so that $x_b = f_b x_b$. Further in |7| it is proved that for an arbitrary $y_o^a \in M$, $x_a = \lim_{n\to\infty} y_n^a$, where $y_n^a = f_a y_{n-1}^a$ $(n \ge 1)$ and that:

$$p_i(y_{k+1}^a - y_k^a) \le \prod_{s=o}^{k-1} q_{g^s(i)}(p_{g^{s+1}(i)}(y_{k-s}^a - y_{k-s-1}^a)) p_{g^k(i)}(y_o^a - y_1^a)$$

for every $a \in A$, $k \in N$ and $i \in I$.

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Since $p_{g^k(i)}(y_o^a - y_1^a) \leq h_i(y_o^a - y_1^a)$ for every $k \in N \cup \{0\}$, $a \in A$ and $i \in I$ and $q_i(t) \in [0,1]$ for every $i \in I$ and every $t \in \mathbb{R}^+$ it follows that:

$$p_j(y_n^a - y_{n-1}^a) \leq h_i(y_o^a - y_1^a)$$

for every $n \in N$, $a \in A$ and $j \in O(i,g) = \{i, g(i), g^2(i), \ldots\}$.

So we have:

$$p_i(y_{k+1}^a - y_k^a) \leq \prod_{s=o}^{k-1} q_{g^s(i)}(h_i(y_o^a - y_1^a))h_i(y_o^a - y_1^a), \quad i \in I, \quad a \in A.$$

From b) it follows that y_o^a can be an arbitrary element from M and so let $y_o^a = x_o$, for every $a \in A$. Then:

$$p_i(y_{k+1}^a - y_k^a) \leq \prod_{s=o}^{k-1} q_{g^s(i)}(h_i(x_o - f_a x_o))h_i(x_o - f_a x_o), \quad k \in N, \ i \in I, \ a \in A.$$

Further, it follows that the set $\{x_o - f_a x_o \mid a \in A\}$ is bounded and by using b) we conclude that there exists $N_i \in \mathbb{R}^+$ so that

$$h_i(x_o - f_a x_o) \leq N_i, \quad \text{for every } a \in A \ (i \in I)$$

which implies that:

$$(2) \quad p_i(y_{k+1}^a - y_k^a) \leq \prod_{s=o}^{k-1} q_{g^s(i)}(N_i)h_i(x_o - f_a x_o) \quad (k \in N, \ a \in A, \ i \in I).$$

From (2) we have:

$$p_i(y_m^a - y_n^a) \leq \sum_{s=n}^{m-1} p_i(y_{s+1}^a - y_s^a) \leq \sum_{s=n}^{m-1} \left(\prod_{k=o}^{s-1} q_{g^k(i)}(N_i)\right)h_i(x_o - f_a x_o)$$

and when $m \to \infty$.

$$(3) \quad p_i(x_a - y_n^a) \leq \sum_{s=n}^{\infty} \left(\prod_{k=o}^{s-1} q_{g^k(i)}(N_i)\right)h_i(x_o - f_a x_o) \ (n \in N, \ i \in I, a \in A).$$

Since $\overline{\lim_{n\to\infty}} q_{g^n(i)}(t) < 1$ it follows that the series $\sum_{s=1}^{\infty} \left(\prod_{k=o}^{s-1} q_{g^k(i)}(N_i)\right)$ is convergent for every $i \in I$. From (3) we have:

$$p_i(x_a - y_1^a) \leq \sum_{s=1}^{\infty} \left(\prod_{k=o}^{s-1} q_{g^k(i)}(N_i)\right)h_i(x_o - f_a x_o)$$

and so:

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$$p_i(x_a - x_o) \leq p_i(x_a - f_a x_o) + p_i(f_a x_o - x_o) \leq \sum_{s=1}^{\infty} \left(\prod_{k=o}^{s-1} q_{g^k(i)} (N_i) \right) \times$$

$$\times\, h_i(x_o - f_a x_o) + p_i(f_a x_o - x_o) \quad (i \in I,\ a \in A).$$

Since $x_o - f_a x_o = f_o x_o - f_a x_o$ and $\lim_{a \in A} f_o x_o - f_a x_o = 0$, by using the continuity of h_i and $h_i(0) = 0$, it follows that:

$$\lim_{a \in A} p_i(x_a - x_o) = 0, \quad \text{for every } i \in I$$

which means that $\lim_{a \in A} x_a = x_o$.

In the following Lemma, E is as in Lemma 1. Lemma 2 is a generalization of Theorem 1 from |6|, for bounded T(M).

LEMMA 2. *Let* M *be an arbitrary set,* T *be a transformation from* M *into* E *such that* T(M) *is a sequentially complete, bounded closed set and let* $\{g_a\}_{a \in A}$ *be a net of transformations defined on* M *with the values in* E *and* $g_a(M) \subseteq T(M)$, *for all* $a \in A$. *Suppose that for every* $a \in A$ *and every* $x \in M$ *there exists* $\lim_{a \in A} g_a x = g_o x$. *Further, for every* $i \in I$ *there exists* $f(i) \in I$ *and* $q_i : \mathbb{R}^+ \to [0,1]$ *as in Lemma 1 so that:*

$$p_i(g_a x - g_a y) \leq q_i(p_{f(i)}(Tx - Ty))\, p_{f(i)}(Tx - Ty)$$

for every $x, y \in M$ *and condition* b) *of Lemma 1 for* $f = g$ *is satisfied. Let* $b \in A \cup \{0\}$. *Then:*

1. *For every* $y \in T(M)$ *the set* $g_b(T^{-1}y)$ *has only one element and the mapping* $f_b : y \to g_b(T^{-1}y)$ *has one and only one fixed point* $y_b \in T(M)$.

2. *For every* $x \in T^{-1}y$, $g_b x = Tx$ *and* $g_b x^{(i)} = Tx^{(i)}$ $(i \in \{1,2\})$ *implies that* $Tx^{(1)} = Tx^{(2)}$.

3. $\lim_{a \in A} Tx_a = Tx_o$ $(x_b \in T^{-1}y_b,\ b \in A \cup \{0\})$.

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P r o o f. First, we shall prove that $g_b(T^{-1}y)$ contains only one element. Let $y \in T(M)$ and $Tx_i = y$ $(i \in \{1,2\})$. We shall prove that $v_1 = v_2$, where $v_i = g_b x_i$, $i \in \{1,2\}$. We have:

$$p_i(v_1 - v_2) = p_i(g_b x_1 - g_b x_2) \leq q_i(p_{f(i)}(Tx_1 - Tx_2)) p_{f(i)}(Tx_1 - Tx_2).$$

Since $Tx_1 = Tx_2 = y$ it follows that $v_1 = v_2$. From this we conclude that f_b is a singlevalued mapping. Since $g_b(M) \subseteq T(M)$ it follows that $f_b : T(M) \to T(M)$. Further let $x, y \in T(M)$, $x = Tu$, $y = Tv$. Then:

$$p_i(f_b x - f_b y) = p_i(g_b u - g_b v) \leq q_i(p_{f(i)}(Tu - Tv)) p_{f(i)}(Tu - Tv) =$$

$$= q_i(p_{f(i)}(x - y)) p_{f(i)}(x - y)$$

and so for every $b \in A \cup \{0\}$ there exists one and only one element $y_b \in T(M)$ so that $f_b y_b = g_b(T^{-1}y_b) = y_b$. Applying Lemma 1 we conclude that $\lim_{a \in A} y_a = y_o$. The proofs of 2. and 3. are as in Theorem 1 |6|.

THEOREM 1. *Let $M \subseteq E, K$ be a closed and convex subset of E, $T : M \to K, T(M)$ be complete and bounded, $Q : K \to E$ be a compact mapping, $F : M \times K \to T(M)$ so that the following conditions are satisfied:*

1. *For every $i \in I$ there exist $f(i) \in I$ and $q_i : \mathbb{R}^+ \to [0,1]$ as in Lemma 1 so that:*

$$p_i(F(x_1, y) - F(x_2, y)) \leq q_i(p_{f(i)}(Tx_1 - Tx_2)) p_{f(i)}(Tx_1 - Tx_2)$$

for every $x_1, x_2 \in M$ and $y \in K$ and the condition b) of Lemma 1 is satisfied for $g = f$.

2. *For every $i \in I$ there exists $C(i) > 0$ so that:*

$$p_i(F(x, y_1) - F(x, y_2)) \leq C(i) p_i(Qy_1 - Qy_2)$$

for every $x \in M$ and every $y_1, y_2 \in K$ and for every $i \in I$ the serie :

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$$\sum_{k=1}^{\infty} \Big(\prod_{r=0}^{k-1} q_{f^r(i)}(t)\Big) C(f^k(i))$$

is convergent, for every $t \in \mathbb{R}^+$.

Then there exists $x \in M$ *so that* $F(x,Tx) = Tx$.

P r o o f. From Lemma 2 it follows that for every $y \in K$ there exists $u(y) \in M$ such that $F(u(y),y) = Tu(y)$. If $\tilde{f} : y \mapsto Tu(y)$ $(y \in K)$ then $\tilde{f}(K) \subseteq K$. The continuity of $\tilde{f}$ follows as in |6| since all the conditions of Lemma 2 are satisfied, where $\{x_a\}_{a\in A}$ is a convergent net in K, $\lim_{a\in A} x_a = x_o$ and $g_a x = F(x,x_a)$, $g_o x = F(x,x_o)$, for $x \in M$. Let us prove that $\overline{\tilde{f}(K)}$ is compact which means that for every net $\{x_a\}_{a\in A}$ in K there exists a convergent subnet of the net $\{fx_a\}_{a\in A} = \{F(u_{x_a},x_a)\}_{a\in A}$. We have $(a,c \in A)$:

$$p_i(F(u_{x_a},x_a) - F(u_{x_c},x_c)) \le p_i(F(u_{x_a},x_a) - F(u_{x_c},x_a)) +$$

$$+ p_i(F(u_{x_c},x_a) - F(u_{x_c},x_c)) \le q_i(p_{f(i)}(Tu_{x_a} - Tu_{x_c}))p_{f(i)}(Tu_{x_a} - Tu_{x_c})$$

$$+ C(i)p_i(Qx_a - Qx_c) \le q_i(p_{f(i)}(Tu_{x_a} - Tu_{x_c}))\big[q_{f(i)}(p_{f^2(i)}(Tu_{x_a} - Tu_{x_c})$$

$$\times p_{f^2(i)}(Tu_{x_a} - Tu_{x_c}) + C(f(i))p_{f(i)}(Qx_a - Qx_c))\big] + C(i)p_i(Qx_a - Qx_c) \le$$

$$\le \dots \le \prod_{s=0}^{n} q_{f^s(i)}(p_{f^{s+1}(i)}(Tu_{x_a} - Tu_{x_c}))p_{f^{n+1}(i)}(Tu_{x_a} - Tu_{x_c}) +$$

$$+ \sum_{k=0}^{n-1} \Big(\prod_{s=0}^{k} q_{f^s(i)}(p_{f^{s+1}(i)}(Tu_{x_a} - Tu_{x_c}))\Big) C(f^{k+1}(i))p_{f^{k+1}(i)}($$

$(Qx_a - Qx_c) + C(i)p_i(Qx_a - Qx_c)$. Further there exists $M(i) > 0$ $(i \in I)$ so th

$$p_{f^{s+1}(i)}(Tu_{x_a} - Tu_{x_c}) \le M(i) \quad (s \in N \cup \{0\},\ i \in I,\ a,c \in A).$$

Then:

$$p_i(F(u_{x_a},x_a) - F(u_{x_c},x_c)) \le \prod_{s=0}^{n} q_{f^s(i)}(M(i))M(i) + C(i)p_i(Qx_a - Qx_c) +$$

$$\sum_{k=0}^{n-1} \Big(\prod_{s=0}^{k} q_{f^s(i)}(M(i))C(f^{k+1}(i))h_i(Qx_a - Qx_b).$$

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If $n \to \infty$ then:

$$p_i(F(u_{x_a},x_a)-F(u_{x_c},x_c)) \leq C(i)p_i(Qx_a-Qx_c) + \sum_{r=o}^{\infty} \left(\prod_{s=o}^{r} q_{f^s(i)}(M(i))\right) \cdot$$

$$\cdot C(f^{r+1}(i))h_i(Qx_a-Qx_c) .$$

The set $\overline{Q(K)}$ is compact and so there exists a convergent subnet $\{Qx_a\}_{a\in A'}$ of the net $\{Qx_a\}_{a\in A}$. Now, we shall show that $\{F(u_{x_a},x_a)\}_{a\in A'}$ is a Cauchy net which means that there exists $a_o \in A'$, such that $a,c \geq a_o(i,\varepsilon)$ implies $(\varepsilon > 0)$:

$$p_i(F(u_{x_a},x_a) - F(u_{x_c},x_c)) < \varepsilon .$$

Since h_i is continuous and $h_i(0) = 0$ it follows that for every $\varepsilon' > 0$ there exists a neighbourhood $V_{i,\varepsilon'}(0)$ of zero so that

$$z \in V_{i,\varepsilon'} \Longrightarrow h_i(z) < \frac{\varepsilon'}{K(i)}$$

where $K(i) = \sum_{r=o}^{\infty} \left(\prod_{s=o}^{r} q_{f^s(i)}(M(i))C(f^{r+1}(i))+C(i)\right)$. Since $\{Qx_a\}_{a\in A'}$ is convergent there exists $a_o(i,\varepsilon)$ so that $a,c \geq a_o$ $(a,c \in A')$ implies:

$$Qx_a - Qx_c \in V_{i,\varepsilon} .$$

Then we have that $a,c \geq a_o$ (i,ε) $(a,c \in A')$ implies that:

$$p_i(F(u_{x_a},x_a) - F(u_{x_c},x_c)) < \varepsilon.$$

The rest of the proof is as in |6|.

As in |6| and using a similar method as in Theorem 1 we can prove the following theorem.

THEOREM 2. *Let $M \subseteq E, K$ be a closed and convex subset of E, $T : M \to E, \overline{T(M)}$ be bounded complete $T(M) \subseteq K, Q : K \to E$ be a compact mapping and $F : M \times \overline{Q(K)} \to E$ so that for every $x \in M$ the mapping*

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$y \to F(x,y)$ $(y \in \overline{QK})$ *is continuous,* $F(M,\overline{Q(K)}) \subseteq T(M)$ *and the following conditions are satisfied:*

For every $i \in I$ *there exist* $f(i) \in I$ *and* $q_i : \mathbb{R}^+ \to [0,1)$ *as in Lemma 1 so that:*

$$p_i(F(x_1,y)-F(x_2,y)) \leq q_i(p_{f(i)}(Tx_1-Tx_2))p_{f(i)}(Tx_1-Tx_2)$$

for every $x_1,x_2 \in M$ *and every* $y \in \overline{Q(K)}$ *and condition* b) *of Lemma 1 is satisfied for* $g = f$.

Then there exists $x \in M$ *so that* $F(x, Q(Tx)) = Tx$.

From Theorem 2 we obtain the following Corollary, which is a generalization of Krasnoseljski´s fixed point theorem.

COROLLARY *Let* $K=\bar{K}$ *be a bounded, convex and complete subset of* E, $Q : K \to E$ *be a compact mapping,* $F : K \times \overline{Q(K)} \to E$ *so that for every* $x \in K$ *the mapping* $y \to F(x,y)$ $(y \in \overline{Q(K)})$ *is continuous,* $F(K,\overline{Q(K)}) \subseteq K$ *and the following condition is satisfied:*

For every $i \in I$ *there exist* $f(i) \in I$ *and* $q_i : \mathbb{R}^+ \to [0,1)$ *as in Lemma 1 so that:*

$$p_i(F(x_1,y)-F(x_2,y)) \leq q_i(p_{f(i)}(x_1-x_2))p_{f(i)}(x_1-x_2)$$

for every $x_1,x_2 \in K$ *and every* $y \in \overline{Q(K)}$ *and condition* b) *of Lemma 1 is satisfied for* $g = f$..

Then there exists $x \in K$ *so that* $F(x,Qx) = x$.

P r o o f. It is enough to take that $T = IdK$, $M = K$.

REMARK. In Krasnoseljski´s fixed point theorem is $F(x,y) = Sx + y$, where S is a contraction type mapping.

Using a similar method as in |6| we can prove the following Lemma.

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LEMMA 3. *Let* $(E, \{p_i\}_{i\in I})$ *be a Hausdorff locally convex topological vector space in which the topology is defined by the family* $\{p_i\}_{i\in I}$ *od seminorm,* $g : I \to I$, M *be a closed sequentially complete subset of* E, *for every* $a \in A$, $f_a : M \to M$ *so that* $\lim_{a\in A} f_a x$ *exists for every* $x \in M$ *and the following conditions are satisfied:*

1. *For every* $i \in I$ *there exists* $q(i) > 0$ *so that:*

$$p_i(f_a x - f_a y) \leq q(i) p_{g(i)}(x-y),$$ *for every* $x,y \in M$ *and every* $a \in A$.

2. *For every* $i \in I$ *and every* $n \in N$ *there exist* $a_n(i) \geq 0$ *and* $f(i) \in I$ *so that* $p_{g^n(i)}(x) \leq a_n(i) p_{f(i)}(x)$, *for every* $x \in E$ *and the series:*

$$\sum_{n=1}^{\infty} \left(\prod_{r=0}^{n-1} q(g^r(i)) \right) a_n(i)$$

is convergent for every $i \in I$.

If $f_o x = \lim_{a\in A} f_a x$, *for every* $x \in M$ *then* $\lim_{a\in A} x_a = x_o$, *where* $f_a x_a = x_a$, *for every* $a \in A \cup \{0\}$.

Using Lemma 3 instead of Lemma 1 it is easy to prove the following theorem on the existence of a solution of the equation $Tx = F(x,Q(Tx))$.

THEOREM 3. *Let* $E, M, K, T, \overline{T(M)}, Q$ *and* F *be as in Theorem 2 and the following condition be satisfied:*

For every $i \in I$ *there exist* $q(i) > 0$ *and* $g(i) \in I$ *so that:*

$$p_i(F(x_1,y) - F(x_2,y)) \leq q(i) p_{g(i)}(Tx_1 - Tx_2)$$

for every $x_1, x_2 \in M$ *and every* $y \in \overline{Q(K)}$ *and condition 2. of Lemma 3 is satisfied.*

Then there exists $x \in M$ *so that* $F(x,Q(Tx)) = Tx$.

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Suppose that E is a vector space in which the topology is defined by the family $\{p_i\}_{i\in I}$ of functionals such that the following conditions are satisfied:

1. $p_i(0) = 0$, for every $i \in I$.
2. $p_i(-x) = p_i(x)$, for every $x \in E$ and every $i \in I$.
3. $p_i(x+y) \leq p_i(x)+p_i(y)$, for every $x,y \in E$ and every $i \in I$.
4. If $p_i(x_a-x) \to 0$, ($\{x_a\}$ is a net from E) and $t_a \to t$ then $p_i(t_a x_a - tx) \to 0$, $a \in A$.

Then E is a topological vector space in which the fundamental system of neighbourhoods of zero in E is given by the family:

$$V(i,r) = \{x \mid p_i(x) < r\} \quad (i \in I,\ r > 0).$$

Suppose now that $(E, \{p_i\}_{i\in I})$ is a topological vector space in which the topology is defined by the family of functionals $\{p_i\}_{i\in I}$ so that 1.,2.,3. and 4. are satisfied. We shall suppose that E is Hausdorff which means that $p_i(x) = 0$, for every $i \in I$ implies that $x = 0$.

Similarly as in |8|, it is easy to prove the following fixed point theorem.

THEOREM 4. *Let* $(E, \{p_i\}_{i\in I}$ *be a Hausdorff topological vector space in which the topology is defined by the family of functionals* $(i \in I)$ *so that* 1.,2.,3. *and* 4. *is satisfied. Let* M *be a closed and convex subset of* E *and* $F : M \to M$ *be a compact mapping such that the following condition is satisfied:*

For every $i \in I$ *there exists* $C_i > 0$ *so that:*

$p_i(tx) \leq C_i t p_i(x)$, *for every* $t \in [0,1]$ *and* $x \in F(M) - F(M)$.

Then there exists $x \in M$ *so that* $x = Fx$.

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P r o o f. First, we shall show that for every V from the fundamental system of neighbourhoods of zero in E there exists a neighbourhood of zero U so that:

$$co(U \cap (F(M) - F(M)) \subseteq V$$

where co is convex hull. Let $V = V(i,r)$ $(i \in I,\ r > 0)$. Then $U = V(i, \frac{r}{C_i})$. Indeed, suppose that $z \in co(U \cap (F(M)-F(M))$. Then there exist $s_i \geq 0$ $(i \in \{1,2,\dots,n\})$ so that $\sum_{i=1}^{n} s_i = 1$ and $u_i \in U \cap (F(M)-F(M))$ $(i \in \{1,2,\dots,n\})$ so that:

$$z = \sum_{i=1}^{n} s_i u_i .$$

Then we have:

$$p_i(z) \leq \sum_{j=1}^{n} p_i(s_j u_j) \leq \sum_{j=1}^{n} C_i s_j p_i(u_j) < \sum_{j=1}^{n} C_i s_j \frac{r}{C_i} = r.$$

Then |5| there exists $x \in M$ such that $x = Fx$.

Now, using Theorem 4 instead of Tihonov's fixed point theorem, it is easy to prove the following generalization of Theorem 1.

THEOREM 1'. *Let* $(E, \{p_i\}_{i \in I})$ *be a Hausdorff topological vector space in which the topology is defined by the family of functionals* p_i $(i \in I)$ *so that* 1.,2.,3. *and* 4. *are satisfied. Let* M,K,T,Q *and* F *be as in Theorem 1 and conditions*1. *and* 2. *be satisfied as in Theorem 1.Suppose that for every* $i \in I$ *there exists* $C_i > 0$ *so that:*

$p_i(tx) \leq C_i t p_i(x)$, *for every* $t \in [0,1]$ *and every* $x \in T(M)-T(M)$. *Then there exists* $x \in M$ *so that* $F(x,Tx) = Tx$.

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REZIME

O JEDNAČINI Tx=(Fx,Q(Tx)) U LOKALNO KONVEKSNIM PROSTORIMA

U ovom radu su, korišćenjem teorema o nepokretnoj tački iz rada |4| i |7|,dokazana neka uopštenja teorema o egzistenciji rešenja jednačina Tx=F(x,Tx), i Tx=F(x,Q(Tx)) u lokalno konveksnim prostorima koje su dokazane u radu |6|.

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A FIXED POINT THEOREM FOR MULTIVALUED QUASI-CONTRACTIONS IN PROBABILISTIC METRIC SPACES

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ABSTRACT

In this note, we first generalize Minh's theorem for spaces with a family of pseudo-metrics, next we introduce the notion of the probabilistic largest distance between two sets and establish some of its simple properties, and finally, we prove an analogue to Minh's theorem for PM-spaces.

1. INTRODUCTION

In |1| Ćirić has proved a fixed point theorem for multivalued quasi-contractions, i.e. for mappings satisfying the following condition

$$(1)\quad \rho(Tx,Ty) \leq k.\max\{d(x,y),\ \rho(x,Tx),\ \rho(y,Ty),\ d(x,Ty),\ d(y,Tx)\}$$

where $k < 1$, d is a metric defined in X, ρ stands for the largest distance between two sets defined by

$$\rho(A,B) = \sup\{d(x,y): x \in A,\ y \in B\}$$

and finally, as usual,

$$d(x,A) = \inf\{d(x,y): y \in A\}.$$

Then, in |2| N.A. Minh generalized Ćirić's result by showing that in (1) all the d's can be replaced by ρ.

Note that up to now there have been a lot of fixed point theorems concerning multivalued mappings of the contractiv

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type in metric spaces, but none of their analogues in probabilistic matric spaces (in abbreviation, PM-spaces) has appeared even for the simplest case such as for Nadler's theorem |3|. The main difficulty is that the topology in a PM-space depends on a parameter $\lambda \in (0,1)$, so we cannot choose an iterative sequence independent of λ without additional assumptions. Recently on the way to getting an analogue to Nadler's theorem in PM-spaces Hadžić proved a theorem whose assumptions assure the existence of such a sequence |4|.

2. MULTIVALUED QUASI-CONTRACTIONS IN SPACES WITH A FAMILY OF PSEUDO-METRICS

First let us recall the following definition. Let X be an arbitrary set, a mapping $d : X \times X \to R_+$ (the set of all non-negative numbers) is called a pseudo-metric if it satisfies the following conditions

$$d(x,x) = 0,$$
$$d(x,y) = d(y,x),$$
$$d(x,y) \leq d(x,z) + d(z,y)$$

for every x,y,z in X.

Let Λ be an index set. A pair (X,d_λ), $\lambda \in \Lambda$, is called a space with a family of pseudo-metrics if d_λ is a pseudo-metric for each $\lambda \in \Lambda$. In the sequel we always assume that the family $\{d_\lambda\}$ satisfies the following "separation condition"

$$d_\lambda(x,y) = 0 \quad (\forall \lambda \in \Lambda) \Rightarrow x = y .$$

In |5| such a space is called uniformizable and some fixed point theorems for singlevalued mappings of the contractive type in these spaces are established.

The notions and results in this section mostly repeat that of |2| for each fixed λ , but for convenience to readers we shall develop them here briefly (the proof is partially similar to that of |1|).

DEFINITION 2 . *Let Y be an arbitrary set. A mapping* $\rho : X \times X \to R_+$ *is called a semimetric if it satisfies the following conditions*

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$$\rho(x,y) = \rho(y,x),$$
$$\rho(x,y) \leq \rho(x,z) + \rho(z,y)$$

for every x,y,z *in* Y.

Let (Y,ρ_λ) be a space with a family of semimetrics $\{\rho_\lambda : \lambda \in \Lambda\}$. A sequence $\{x_n\} \subseteq Y$ is called a Cauchy sequence if $\lim_{n,m\to\infty} \rho_\lambda(x_n,x_m) = 0$ for each λ. A sequence $\{x_n\}$ is said to be convergent to $x \in Y$ (symbolically, $x_n \to x$) if $\lim_{n\to\infty} \rho_\lambda(x_n,x) = 0$ for each λ. The space Y is said to be complete if every Cauchy sequence is convergent. Note that these notions are formal because a semimetric does not generate any topology, this is easily seen when we consider a stationary sequence $x_n \equiv x$ with $\rho(x,x) > 0$.

In what follows we assume that the family of semimetrics $\{\rho_\lambda\}$ has the following property

$$\rho_\lambda(x,y) = 0 \quad (\forall \lambda \in \Lambda) \implies x = y.$$

Obviously each pseudo-metric is a semimetric. The largest distance between two bounded sets in a metric space is a nontrivial example of semimetrics.

LEMMA 1. *Let* (Y,ρ_λ) *be a complete space with a family of semimetrics,* $T : Y \to Y$ *a mapping satisfying the following condition: for each* $\lambda \in \Lambda$ *there is a* $q_\lambda < 1$ *such that*

$$\rho_\lambda(Tx,Ty) \leq q_\lambda \cdot \max\{\rho_\lambda(x,y), \rho_\lambda(x,Tx), \rho_\lambda(y,Ty), \rho_\lambda(x,Ty), \rho_\lambda(y,Tx)\} \tag{2}$$

for every x,y *in* Y.

Then there exists a unique fixed point x^* *of* T. *Moreover, we have* $\rho_\lambda(x^*,x^*) = 0$ *for each* $\lambda \in \Lambda$ *and* $T^n x \to x^*$ *for every* x *in* Y.

P r o o f. We shall use the following notations

$$\theta(x,n) = \{T^i x : i = 0,1,\ldots,n\}, \text{ where } T^0 x = x,$$
$$\theta(x,\infty) = \{T^i x : i = 0,1,2,\ldots\},$$
$$\delta_\lambda(A) = \sup\{\rho_\lambda(x,y) : x,y \in A\}.$$

Fixing $\lambda \in \Lambda$ we have

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a) if $1 \leq i,j \leq n$ then $\rho_\lambda(T^i x, T^j x) \leq q_\lambda \delta_\lambda(\theta(x,n))$. From this it follows that $\delta_\lambda(\theta(x,n)) = \rho_\lambda(x,T^k x)$ with some $k \leq n$.

b)
$$\delta_\lambda(\theta(x,n)) \leq \max\{\frac{\rho_\lambda(x,Tx)}{1-q_\lambda}, \rho_\lambda(x,x)\} .$$

Indeed, fixing n, by a) we have $\delta_\lambda(\theta(x,n)) = \rho_\lambda(x,T^k x)$ with some $k \leq n$. If $k \geq 1$ then

$$\delta_\lambda(\theta(x,n)) = \rho_\lambda(x,T^k x) \leq \rho_\lambda(x,Tx) + \rho_\lambda(Tx,T^k x) \leq$$

$$\leq \rho_\lambda(x,Tx) + q_\lambda \delta_\lambda(\theta(x,n)) .$$

From this b) follows.

c)
$$\rho_\lambda(T^n x, T^{n+m} x) \leq q_\lambda^n \cdot \max\{\frac{\rho_\lambda(x,Tx)}{1-q_\lambda}, \rho_\lambda(x,x)\} .$$

Indeed, from a) and b) we get

$$\rho_\lambda(T^n x, T^{n+m} x) \leq q_\lambda \delta_\lambda(\theta(T^{n-1}x, m+1)) = q_\lambda \rho_\lambda(T^{n-1}x, T^{n-1+m_1}x)$$

$$\leq q_\lambda^2 \delta_\lambda(\theta(T^{n-2}x, m_1+1)) = \ldots \leq q_\lambda^n \delta_\lambda(\theta(x, m_{n-1}+1)) \leq$$

$$\leq q_\lambda^n \delta_\lambda(\theta(x,\infty)) \leq q_\lambda^n \cdot \max\{\frac{\rho_\lambda(x,Tx)}{1-q_\lambda}, \rho_\lambda(x,x)\} .$$

This implies that $\{T^n x\}$ is a Cauchy sequence and hence $T^n x \to x^*$. Since $\rho_\lambda(x^*,x^*) \leq \rho_\lambda(x^*,T^n x) + \rho_\lambda(T^n x, x^*)$, we get $\rho_\lambda(x^*,x^*) = 0$.

d) We have

$$\rho_\lambda(x^*,Tx^*) \leq \rho_\lambda(x^*,T^{n+1}x) + \rho_\lambda(T^{n+1}x,Tx^*) \leq \rho_\lambda(x^*,T^{n+1}x) +$$

$$+ q_\lambda \max\{\rho_\lambda(T^n x,x^*), \rho_\lambda(T^n x,T^{n+1}x), \rho_\lambda(x^*,Tx^*),$$

$$\rho_\lambda(T^n x,Tx^*), \rho_\lambda(x,T^{n+1}x)\} .$$

Letting $n \to \infty$ we obtain

$$\rho_\lambda(x^*,Tx^*) \leq q_\lambda \rho_\lambda(x^*,Tx^*) ,$$

from this $\rho_\lambda(x^*,Tx^*) = 0$ for each λ and hence $x^* = Tx^*$. The uniqueness of x^* is proved in the usual way and the lemma follows.

Let (X,d_λ) be a space with a family pseudo-metrics. By ρ_λ we denote the largest distance between two sets generated

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by d_λ, $\delta_\lambda(A) = \rho_\lambda(A,A)$ - the λ-diameter of A, B(X) - the class of all non-empty subsets of X with finite λ-diameter for each λ (these subsets are said to be bounded).

LEMMA 2. *If* (X,d_λ) *is a space with a family of pseudo-metrics then* $(B(X),\rho_\lambda)$ *is a space with a family of semimetrics. Moreover, if* X *is complete, so is* B(X).

P r o o f. The symmetry and the triangle inequality of ρ_λ follow immediately from the corresponding properties of d_λ. Further, if $\rho_\lambda(A,B) = 0$ for each λ then we have $d_\lambda(x,y)=0$ for each λ and for every $x \in A$, $y \in B$, it, in turn, implies that $x = y = A = B$. Now, let $\{A_n\}$ be a Cauchy sequence in B(X). Taking $x_n \in A_n$ for each n we obtain a Cauchy sequence $\{x_n\}$ in X. By the completeness of X, $x_n \to x$. Fixing $\lambda \in \Lambda$ we get

$$\rho_\lambda(x,A_n) \le \rho_\lambda(x,x_n) + \rho_\lambda(x_n,A_n) \le d_\lambda(x,x_n) + \rho_\lambda(A_n,A_n) .$$

Letting $n \to \infty$ we get $\lim_{n\to\infty} \rho_\lambda(x,A_n) = 0$ for each λ, i.e. by definition $A_n \to \{x\}$ in B(X). So B(X) is complete and the lemma follows.

THEOREM 1. *Let* (X,d_λ) *be a complete space with a family of pseudo-metrics,* $T : X \to B(X)$ *be a mapping satisfying the following condition: for each* $\lambda \in \Lambda$ *there is a* $q_\lambda < 1$ *such that*

$$(3) \quad \rho_\lambda(Tx,Ty) \le q_\lambda \max\{d_\lambda(x,y), \rho_\lambda(x,Tx), \rho_\lambda(y,Ty), \rho_\lambda(x,Ty), \rho_\lambda(y,Tx)\}$$

for every x,y *in* X .

Then T *has a unique fixed point* x*. *Moreover, in fact, it is a stationary point (i.e.* Tx* = {x*}).

P r o o f. First we show that T maps B(X) into B(X). Indeed, let $A \in B(X)$ fix $x_o \in A$ and take an arbitrary x in A. Then we have

$$\rho_\lambda(Tx,Tx_o) \le q_\lambda \max\{d_\lambda(x,x_o), \rho_\lambda(x,Tx), \rho_\lambda(x_o,Tx_o), \rho_\lambda(x,Tx_o), \rho_\lambda(x_o,Tx)\} .$$

Suppose for example, the maximum is attained at $\rho_\lambda(x,Tx)$. Then we have

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$$\rho_\lambda(Tx,Tx_o) \le q_\lambda \rho_\lambda(x,Tx) \le q_\lambda \rho_\lambda(x,Tx_o) + q_\lambda \rho_\lambda(Tx,Tx_o) .$$

Consequently

$$\rho_\lambda(Tx,Tx_o) \le \frac{q_\lambda}{1-q_\lambda} \rho_\lambda(A,Tx_o) .$$

Similarly for the case when the maximum is attained at $\rho_\lambda(x_o,Tx)$ and thus we get

$$\rho_\lambda(TA,Tx_o) \le \max\{q_\lambda \delta_\lambda(A), \frac{q_\lambda}{1-q_\lambda} \rho_\lambda(A,Tx_o)\} .$$

From this $TA \in B(X)$.

Obviously, from (3) we have for every A,B in B(X)

$$\rho_\lambda(TA,TB) \le q_\lambda \max\{\rho_\lambda(A,B), \rho_\lambda(A,TA), \rho_\lambda(B,TB),$$
$$\rho_\lambda(A,TB), \rho_\lambda(B,TA)\} .$$

From Lemmas 1 and 2 it follows that there is a unique $A_o \in B(X)$ such that $TA_o = A_o$, moreover $\rho_\lambda(A_o,A_o) = 0$ for each λ , i.e. A_o is a singleton $\{x^*\}$.

The uniqueness of x^* is proved as follows. Let y^* be another fixed point of T, i.e. $y^* \in Ty^*$, By (3) we have

$$\rho_\lambda(Ty^*,Ty^*) \le q_\lambda \max\{d_\lambda(y^*,y^*), \rho_\lambda(y^*,Ty^*)\} \le q_\lambda \rho_\lambda(Ty^*,Ty^*).$$

From this $\rho_\lambda(Ty^*,Ty^*) = 0$, it implies that $Ty^* = \{y^*\}$ and $\rho_\lambda(y^*,y^*) = 0$. Now again by (3) we get

$$d_\lambda(x^*,y^*) = d_\lambda(Tx^*,Ty^*) \le q_\lambda d_\lambda(x^*,y^*)$$

for each λ , hence $x^* = y^*$. The proof of the theorem is complete.

3. THE PROBABILISTIC LARGEST DISTANCE BETWEEN TWO SETS.

Let us recall some definitions. A function $F : R \to [0,1]$ (here R denotes the set of all real numbers) is called a distribution function if it is non-decreasing, left-continuous, inf F = 0 and sup F = 1. The family of all distribution functions is denote by $\mathcal{D}$. A PM-space is a pair (X,F), where X is a certain set and $F : X \times X \to \mathcal{D}$ satisfying the following condition:

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(here instead of $F(x,y)$ we write F_{xy} for simplicity of notation)

$$F_{xy}(t) = 1 \quad (\forall t > 0) \iff x = y ,$$

$$F_{xy}(0) = 0,$$

$$F_{xy} = F_{yx} ,$$

$$F_{xy}(t+s) \geq \min \{ F_{xz}(t), F_{zy}(s) \}$$

for every x,y,z in X and $t,s > 0$ (see |7| for details).

Similarly to the notion of probabilistic diameter introduced in |6|, here we define

$$\delta_{AB}(t) = \sup_{s<t} \inf_{x \in A, y \in B} F_{xy}(s) ,$$

where A and B are probabilistic bounded sets, i.e. they satisfy the condition $\sup_{t>0} D_C(t) = 1$ with $D_C(t) = \delta_{CC}(t)$.

Here are some simple properties of δ_{AB}.

1. δ_{AB} is a distribution function.

Evidently, δ_{AB} is non-decreasing and left-continuous, $\delta_{AB}(0) = 0$. To prove $\sup \delta_{AB} = 1$ we now show that it satisfies the probabilistic triangle inequality, i.e.

$$\delta_{AB}(t+s) \geq \min\{\delta_{AC}(t), \delta_{CB}(s)\}$$

for every A,B,C probabilistic bounded and $t,s > 0$.

Suppose on the contrary that there are A,B,C,t,s such that

$$(4) \qquad a = \delta_{AB}(t+s) < b < \min\{\delta_{AC}(t), \delta_{CB}(s)\} .$$

Since $b < \sup_{v<t} \inf_{x \in A, y \in B} F_{xz}(v)$, there is $v_o < t$ such that $\inf_{x \in A, z \in C} F_{xz}(v_o) \geq b$. Analogously, there exists $w_o < s$ such that $\inf_{z \in C, y \in B} F_{zy}(w_o) \geq b$. By the probabilistic triangle inequality for F_{xy} we have

$$F_{xy}(v_o+w_o) \geq \min \{ F_{xz}(v_o), F_{zy}(w_o) \}$$

for every x,y,z in X. In particular, fixing $x \in A$, $y \in B$ in the left side of the above inequality we obtain

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$$F_{xy}(v_o+w_o) \geq \min\{ \inf_{x\in A, z\in C} F_{xz}(v_o), \inf_{z\in C, y\in B} F_{zy}(w_o)\} \geq b.$$

From this

$$a = \sup_{u<s+t} \inf_{x\in A, y\in B} F_{xy}(u) \geq b$$

and we get a contradiction to (4), so the probabilistic triangle inequality for δ_{AB} is proved.

Now fixing $x_o \in A$, $y_o \in B$ we have

$$\delta_{AB}(t) \geq \min\{\delta_{Ax_o}(\frac{t}{3}), \delta_{x_oy_o}(\frac{t}{3}), \delta_{y_oB}(\frac{t}{3})\} \geq$$

$$\geq \min\{ D_A(\frac{t}{3}), F_{x_oy_o}(\frac{t}{3}), D_B(\frac{t}{3})\} .$$

Since D_A, D_B and $F_{x_oy_o}$ are distribution functions, from this we get the desired result.

2. $\delta_{AB}(t) = 1 \quad (\forall t > 0) \Longrightarrow A = B.$

Indeed, since $\sup_{s<t} \inf_{x\in A, y\in B} F_{xy}(s) = 1$ for each $t > 0$ then for every $\lambda \in (0,1)$ and $t > 0$ there exists $s_o < t$ such that $F_{xy}(s_o) > 1 - \lambda$ for every $x \in A$, $y \in B$. It follows that $F_{xy}(t) = 1$ for every $x \in A$, $y \in B$ and $t > 0$, it in turn implies $A = B$ and moreover, they are a singleton.

By the facts indicated in 1. and 2. and the obvious property $\delta_{AB} = \delta_{BA}$, we can call δ_{AB} a probabilistic semimetric in the space of all probabilistic bounded subsets of X.

3. It is well-known that the (ε,δ)-topology in a PM-space can be described by a family of pseudo-metrics

$$d_\lambda(x,y) = \sup\{t : F_{xy}(t) \leq 1 - \lambda\} \quad , \quad (\lambda \in (0,1)).$$

The λ-larges distance between two bounded sets is defined as usual

$$\rho_\lambda(A,B) = \sup\{d_\lambda(x,y) : x \in A, y \in B\} .$$

Now we are interested in the following problem: In which case have we

(5) $$\rho_\lambda(A,B) = \sup\{t : \delta_{AB}(t) \leq 1 - \lambda\} ? .$$

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First we show that

$$\sup\{t : \delta_{AB}(t) \leq 1-\lambda\} = \sup\{t : \inf_{x\in A, y\in B} F_{xy}(t) \leq 1-\lambda\} .$$

For simplicity of notation we denote for the moment

$$G = \inf_{x\in A, y\in B} F_{xy},\ Z_1 = \{t : G(t) \leq 1-\lambda\},\ Z_2 = \{t : \sup_{s<t} G(s) \leq 1-\lambda\}.$$

So we must prove that $\sup Z_1 = \sup Z_2$.

Since $\sup_{s<t} G(s) \leq G(t)$, we have $\sup Z_1 \leq \sup Z_2$. In order to prove the converse inequality, we take arbitrary $t \in Z_2$ and note that if $s < t$ then $G(s) \leq 1-\lambda$ and hence $s \in Z_1$. This means that $t \leq \sup Z_1$, consequently, $\sup Z_2 \leq \sup Z_1$ as desired.

Thus we now must only answer the question: When does the equality

$$(6)\quad \sup_{x\in A, y\in B} \sup\{t : F_{xy}(t) \leq 1-\lambda\} = \sup\{t : \inf_{x\in A, y\in B} F_{xy}(t) \leq 1-\lambda\}$$

hold ?

Note that the inequality " $\leq$ " always holds since $\inf_{x\in A, y\in B} F_{xy} \leq F_{xy}$ for every $x \in A$, $y \in B$. The equality is not true in general, so we must restrict ourselves to a certain subclass of $B(X)$, namely the class of all nonempty compact (in the (ε,δ)-topology) subsets of X, denoted by $K(X)$.

First we claim that if $A,B \in K(X)$ then $\inf_{x\in A, y\in B} F_{xy}(t) =$

$= \min_{x\in A, y\in B} F_{xy}(t)$, i.e. for each t there are $x_t \in A$, $y_t \in B$ such that $\inf_{x\in A, y\in B} F_{xy}(t) = F_{x_t y_t}(t)$. For this it suffices to verify the lower semi-continuity of $F_{xy}(t)$ considered as a function from $X \times X$ into $[0,1]$ for fixed t. Thus, fixing $(x_o, y_o) \in X \times X$ we have to show that for any $\varepsilon > 0$ there exist a neighbourhood $U_{x_o}(\alpha,\delta)$ of x_o and a neighbourhood $U_{y_o}(\beta,\delta)$ of y_o such that

$$F_{x_o y_o}(t) - \varepsilon \leq F_{xy}(t)$$

for every $x \in U_{x_o}(\alpha,\delta)$ and $y \in U_{y_o}(\beta,\delta)$. Obviously, we may assume

$$\varepsilon < F_{x_o y_o}(t), \quad \text{so} \quad a = F_{x_o y_o}(t) - \varepsilon > 0 .$$

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Suppose on the contrary that for any neighbourhood U of x_o and any neighbourhood V of y_o there exist $\bar{x} \in U$ and $\bar{y} \in V$ such that $a > F_{\bar{x}\bar{y}}(t)$. Letting $\delta = 1 - a$, by the left-continuity of $F_{x_o y_o}$ there is $s < t$ such that $F_{x_o y_o}(s) > a = 1 - \delta$. Taking r so that $s < r < t$ we have

$$F_{xy}(t) \geq \min\{F_{xx_o}(t-r),\ F_{x_o y_o}(s),\ F_{y_o y}(r-s)\}$$

for every $x, y \in X$. Put

$$U_{x_o}(t-r,\delta) = \{x \in X;\ F_{xx_o}(t-r) > 1 - \delta\},$$
$$U_{y_o}(r-s,\delta) = \{y \in X;\ F_{yy_o}(r-s) > 1 - \delta\}$$

and $\bar{x} \in U_{x_o}$, $\bar{y} \in U_{y_o}$ such that $F_{\bar{x}\bar{y}}(t) < a = 1 - \delta$. On the other hand, from the above arguments we get $F_{\bar{x}\bar{y}}(t) > 1 - \delta$. This contradiction proves our assertion.

Now we are in a situation to prove equality (6). Suppose on the contrary that there exists s such that

$$\sup_{x \in A, y \in B} \sup\{t : F_{xy}(t) \leq 1 - \lambda\} < s < \sup\{t : \min_{x \in A, y \in B} F_{xy}(t) \leq 1 - \lambda\}.$$

The first inequality shows that $F_{xy}(s) > 1 - \lambda$ for every $x \in A$, $y \in B$, but this contradicts the second inequality showing that $F_{x_s y_s}(s) \leq 1 - \lambda$ for some $x_s \in A$, $y_s \in B$. Thus, (6) is proved and from this (5) follows.

From (5) and the left-continuity of $\delta_{AB}(t)$ we get an important inequality for our purpose

$$\delta_{AB}(\rho_\lambda(A,B)) \leq 1 - \lambda \tag{7}$$

for every $A, B \in K(X)$.

4. MULTIVALUED QUASI-CONTRACTIONS IN PM-SPACES

Now we can state the main result of this paper.

THEOREM 2. *Let (X, F) be a complete PM-space, $T : X \to K(X)$ a mapping satisfying the following condition: there exists $k < 1$ such that*

$$\delta_{TxTy}(kt) > \min\{F_{xy}(t), \delta_{xTx}(t), \delta_{yTy}(t), \delta_{xTy}(t), \delta_{yTx}(t)\} \tag{8}$$

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for every x,y *in* X, $t>0$.

Then there is a unique fixed point of T , *moreover it is a stationary point.*

P r o o f. Put

$$d_\lambda(x,y) = \sup\{t : F_{xy}(t) \leq 1-\lambda\} \ , \quad (\lambda \in (0,1)).$$

Then (X,d_λ) is a complete space with a family of pseudo-metrics. We have only to show that T satisfies the conditions in Theorem 1.

Suppose the contrary that there are $x,y \in X$ and $\lambda \in (0,1)$ such that

$$\rho_\lambda(Tx,Ty) > k\cdot\max\{d_\lambda(x,y),\rho_\lambda(x,Tx),\rho_\lambda(y,Ty),\rho_\lambda(x,Ty),\rho_\lambda(y,Tx)\}.$$

Put $t = \dfrac{\rho_\lambda(Tx,Ty)}{k}$. Then $t > \max\{d_\lambda(x,y),\ldots,\rho_\lambda(y,Tx)\}$ and by (7) we obtain

$$\min\{F_{xy}(t),\ \delta_{xTx}(t),\ldots,\ \delta_{yTx}(t)\} > 1-\lambda \ .$$

But then from (8) we get

$$\delta_{TxTy}(\rho_\lambda(Tx,Ty)) = \delta_{TxTy}(kt) > 1-\lambda \ ,$$

a contradiction to (7).

Applying Theorem 1, Theorem 2 follows.

Remark that this theorem is a partial analogue of Minh´s theorem in |2|.

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REZIME

TEOREMA O NEPOKRETNOJ TAČKI ZA VIŠEZNAČNE KVAZIKONTRAKCIJE U VEROVATNOSNIM METRIČKIM PROSTORIMA

U ovom radu dokazano je uopštenje teoreme Minha |2| u verovatnosnim metričkim prostorima.

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A FIXED POINT THEOREM IN PROBABILISTIC METRIC SPACES

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ABSTRACT

In this paper a fixed point theorem in probabilistic metric spaces is proved. This theorem is a generalization of a fixed point theorem from [3].

A pair (S,F) is a probabilistic metric space if and only if S is an arbitrary set, $F : S \times S \to \Delta^+$ (Δ^+ is the set of all the distribution functions F such that $F(0) = 0$) so that the following conditions are satisfied ($F(x,y) = F_{x,y}$ for every $x,y \in S$):

1. $F_{x,y}(\varepsilon) = 1$, for every $\varepsilon \in R^+$ iff $x = y$.
2. $F_{x,y} = F_{y,x}$, for every $x,y \in S$.
3. $F_{x,y}(\varepsilon) = 1$ and $F_{y,z}(\delta) = 1$ implies $F_{x,z}(\varepsilon+\delta) = 1$,

where $x,y,z \in S$, $\varepsilon,\delta \in R^+$.

The (ε,λ)-topology in S is introduced by the (ε,λ)-neighbourhoods of $x \in S$:

$$U_x(\varepsilon,\lambda) = \{y : F_{x,y}(\varepsilon) > 1 - \lambda\} \quad \varepsilon > 0, \quad \lambda \in (0,1) .$$

A triplet (S,F,t) is a Menger space iff (S,F) is a probabilistic metric space and t is a T-norm such that for every $\varepsilon_1, \varepsilon_2 \in R^+$

$$F_{x,y}(\varepsilon_1+\varepsilon_2) \geq t(F_{x,z}(\varepsilon_1), F_{z,y}(\varepsilon_2))$$

for every $x,y,z \in S$.

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A set $M \subset S$ is probabilistic bounded iff

$$\sup_{\varepsilon} D_M(\varepsilon) = 1$$

where

$$D_M(\varepsilon) = \sup_{\delta<\varepsilon} \inf_{x,y\in S} F_{x,y}(\delta)$$

is the probabilistic diameter of the set M.

If $f : S \to S$ and $x \in S$ then

$$O_f(x) = \{x, f(x), f^2(x), \ldots\} .$$

DEFINITION 1. |1| *Let* (S,F) *be a probabilistic metric space and* $f : S \to S$. *A point* $x \in S$ *is regular for* f *iff* $\sup_{\varepsilon} D_{O_f(x)}(\varepsilon) = 1$.

DEFINITION 2. |1| *Let* (S,F) *be a probabilistic metric space and* $f : S \to S$. *Two points* $x,y \in S$ *are asymptotic under* f *iff*

$$F_{f^n(x),f^n(y)}(\varepsilon) \to 1 \quad \text{for } n \to \infty , \quad \text{for every } \varepsilon > 0.$$

THEOREM 1. *Let* (S,F,t) *be a Menger space with a continuous* T*-norm* t *and let the mapping* $f : S \to S$ *be such that every point from* S *be regular for* f *and every pair of points from is asymptotic under* f. *Let* $D_{O_f(x)} < D_{O_f(f(x))}$ *whenever* $D_{O_f(x)} < H$. *If for some* $x \in S$ *the sequence* $\{f^n(x)\}_{n\in N}$ *has a convergent subsequence* $\{f^{n_k}(x)\}_{n_k\in N}$ *with a limit a and if*

$$\sup_{\delta<t} \phi(\delta) = \phi(t) ,$$

where

$$\phi(\delta) = \inf_{x,y\in O_f f(a)} F_{x,y}(\delta) ,$$

then the point a *is a unique fixed point of the mapping* f.

P r o o f. First we shall prove that from the conditions of the theorem if follows that

$$F_{a,f^m(a)}(\varepsilon) \geq D_{O_f(f(a))}(\varepsilon)$$

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for all $m \in N$ and for all $\varepsilon > 0$. From the definition of a Menger space we have for all $\delta < \varepsilon$

$$F_{a,f^m(a)}(\varepsilon) \geq t(F_{a,f^{n_k}(x)}(\tfrac{\delta}{2}), F_{f^{n_k}(x),f^m(a)}(\varepsilon - \tfrac{\delta}{2})) \geq$$

$$\geq t(F_{a,f^{n_k}(x)}(\tfrac{\delta}{2}), t(F_{f^{n_k}(x),f^{n_k}(a)}(\tfrac{\delta}{2}), F_{f^{n_k}(a),f^m(a)}(\varepsilon-\delta))).$$

Also we have that

$$F_{f^{n_k}(a),f^m(a)}(\varepsilon-\delta) \geq \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta),$$

which means that

$$F_{a,f^m(a)}(\varepsilon) \geq t(F_{a,f^{n_k}(x)}(\tfrac{\delta}{2}), t(F_{f^{n_k}(x),f^{n_k}(a)}(\tfrac{\delta}{2}),$$

$$\inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))).$$

If $n_k \to \infty$, we have

$$\lim_{n_k\to\infty} F_{a,f^m(a)}(\varepsilon) \geq \lim_{n_k\to\infty} t(F_{a,f^{n_k}(x)}(\tfrac{\delta}{2}), t(F_{f^{n_k}(x),f^{n_k}(a)}(\tfrac{\delta}{2}),$$

$$, \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))).$$

From the preceding inequality we get

$$F_{a,f^m(a)}(\varepsilon) \geq t(H(\tfrac{\delta}{2}), t(H(\tfrac{\delta}{2}), \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))),$$

since

$$\lim_{n_k\to\infty} F_{a,f^{n_k}(x)}(\tfrac{\delta}{2}) = H(\tfrac{\delta}{2}) \quad (\lim_{n_k\to\infty} f^{n_k}(x) = a)$$

$$\lim_{n_k\to\infty} F_{f^{n_k}(x),f^{n_k}(a)}(\tfrac{\delta}{2}) = H(\tfrac{\delta}{2})$$

(x and a are asymptotic points). Therefore, we have

$$F_{a,f^m(a)}(\varepsilon) \geq t(1, t(1, \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))) \geq$$

$$\geq t(1, t(1, \sup_{0<\delta<\varepsilon} \inf_{m,n\in N} F_{f^m(a),f^n(a)}(\varepsilon-\delta))) = D_{O_f(f(a))}(\varepsilon).$$

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So we have proved that

$$F_{a,f^m(a)}(\varepsilon) \geq D_{O_f(f(a))}(\varepsilon) .$$

Now we shall show that

$$D_{O_f(a)}(\varepsilon) = D_{O_f(f(a))}(\varepsilon) .$$

We have

$$D_{O_f(a)}(\varepsilon) = \sup_{t<\varepsilon} \inf_{n,k\in N\cup\{0\}} F_{f^n(a),f^k(a)}(t) =$$

$$\sup_{t<\varepsilon} (\min\{\inf_{n\in N} F_{a,f^n(a)}(t), \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t)\}) .$$

From the next two inequalities

$$F_{a,f^m(a)}(\varepsilon) \geq D_{O_f(f(a))}(\varepsilon) = \inf_{n,k\in N} F_{f^n(a),f^k(a)}(\varepsilon)$$

$$\inf_{n\in N} F_{a,f^n(a)}(t) \geq \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t) ,$$

it follows that

$$\min\{\inf_{n\in N} F_{a,f^n(a)}(t), \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t)\} =$$

$$= \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t)$$

and then we get

$$D_{O_f(a)}(\varepsilon) = \sup_{t<\varepsilon} \inf_{n,k\in N} F_{f^n(a),f^k(a)}(t) = D_{O_f(f(a))}(\varepsilon) .$$

Since $D_{O_f(x)} < D_{O_f(f(x))}$ for all $x \in S$ whenever $D_{O_f(f(x))} < H$, it follows that

$$D_{O_f(a)} = H = D_{O_f(f(a))} .$$

Since

$$F_{a,f^m(a)}(\varepsilon) \geq D_{O_f(f(a))}(\varepsilon) = H(\varepsilon)$$

for all $m \in N$, and every $\varepsilon > 0$ for $m = 1$ we have that

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$$F_{a,f(a)}(\varepsilon) = H(\varepsilon)$$

which means that

$$f(a) = a.$$

If we suppose that a point $b \in S$ is also a fixed point of the mapping f, then we have

$$F_{a,b}(\varepsilon) = F_{f^n(a),f^n(b)}(\varepsilon) \to H(\varepsilon)$$

for $n \to \infty$ (a,b are asymptotic points). From the last equality we get $a = b$.

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REZIME

TEOREMA O NEPOKRETNOJ TAČKI U VEROVATNOSNIM METRIČKIM PROSTORIMA

U ovom radu je dokazana teorema o fiksnoj tački za preslikavanja u verovatnosnim metričkim prostorima. Ona predstavlja generalizaciju teoreme o fiksnoj tački preslikavanja u metričkim prostorima dokazanoj u |3|. Teorem glasi: Neka je (S,F,t) Mengerov prostor sa neprekidnom T-normom t i neka je preslikavanje $f : S \to S$ takvo da je svaka tačka iz S regularna za f i svaki par tačaka iz S je asimptotski u odnosu na f. Neka je $D_{O_f(x)} < D_{O_f(f(x))}$ kadgod je $D_{O_f(x)} < H$. Ako za neko $x \in S$ postoji

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niz $\{f^n(x)\}_{n\in N}$ takav da konvergentni podniz $\{f^{n_k}(x)\}_{n_k\in N}$ ima graničnu vrednost a i ako je

$$\sup_{\delta<t} \phi(\delta) = \phi(t) \ , \quad \text{gde je}$$

$$\phi(\delta) = \inf_{x,y\in O_f(a)} F_{x,y}(\delta) \ ,$$

tada je tačka a jedinstvena nepokretna tačka preslikavanja f.

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ON MULTIFUNCTIONS AND PARACOMPACTNESS

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ABSTRACT

In a paper |15| V.Popa introduced new classes of multifunctions called upper almost continuous (u.a.c.), lower almost continuous (l.a.c.) and almost continuous (a.c.) which are generalizations of upper semicontinuous (u.a.c.), lower semicontinuous (l.s.c.) and continuous multifunctions respectively.

The purpose of the present paper is to investigate some properties of these multifunctions and to obtain a new characterization for paracompact spaces.

1. PRELIMINARIES

DEFINITION 1.1. *A subset of a space is said to be regularly open iff it is the interior of some closed set or equivalently iff it is the interior of its own closure. A set is said to be regularly closed iff it is the closure of some open set or equivalently iff it is the closure if its own interior,* |3|.

A subset is regularly open iff its complement is regularly closed.

DEFINITION 1.2. *A subset* A *of a space* X *is* α*-nearly compact (*N*-closed) iff every* X*-regularly open cover of* A *has a finite subcovering,* |17|.

DEFINITION 1.3. *Let* X *be a topological space and* A *a subset of* X. *The set* A *is* α*-paracompact iff every* X*-open*

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cover of A has an X-open X-locally finite refinement which covers A, |18|.

DEFINITION 1.4. *A space X is paracompact iff every open cover has an open locally finite refinement,* |2|.

In this paper we shall observe multifunctions $F : X \to Y$ such that for each point $x \in X$, $F(x) \neq \emptyset$

DEFINITION 1.5. *The inferior inverse (or briefly, the inverse) of the multifunction* $F:X \to Y$ *is the multifunction denoted by* $F^-:Y \to X$ *and given* $F^-(y) = \{x \in X: y \in F(x)\}$, |1|.

For a multifunction $F : X \to Y$, *we shall denote by* $F^+(G)$ *and* $F^-(G)$ *the upper and lower inverse of the set* $G \subset Y$ *as in* |15|, *and thus*

$$F^+(G) = \{x \in X: F(x) \subset G\};\ F^-(G) = \{x \in X: F(x) \cap G \neq \emptyset\}.$$

DEFINITION 1.6. a) *The multifunction* $F : X \to Y$ *is upper semicontinuous (u.s.c.) (upper almost continuous, u.a.c.) in* $x_o \in X$ *iff for every open set* $V \subset Y$ *containing* $F(x_o)$ *there exists an open set* $U \subset X$ *containing* x_o *such that* $F(x) \subset V$ $(F(x) \subset \subset \alpha(V) = \bar{V}^o)$ *for each* $x \in U$, |1| (|15|).

b) *The multifunction* $F : X \to Y$ *is lower semicontinuous (l.s.c.) (lower almost continuous, l.a.c.) in* $x_o \in X$ *iff for every open set* $V \subset Y$ *with* $V \cap F(x_o) \neq \emptyset$ *there exists an open set U containing* x_o *such that* $F(x) \cap V \neq \emptyset$ $(F(x) \cap \alpha(V) \neq \emptyset)$ *for each* $x \in U$, |1| (|15|).

c) *The multifunction* $F : X \to Y$ *is continous (almost continuous) in* x_o *iff it is both u.s.c. and l.s.c. (u.a.c. and l.a.c.) in* x_o, |1| (|15|).

d) *The miltifunction* $F : X \to Y$ *is u.s.c., l.s.c. and continuous (u.a.c., l.a.c. and almost continuous) iff it has this property in each point* $x \in X$, |1| (|15|).

e) *The multifunction* $F : X \to Y$ *is closed (almost closed) iff for any closed (regularly closed) set* $A \subset X$, $F(A) = \bigcup\{F(x): x \in A\}$ *is closed in Y,* |14|.

f) *The multifunction* $F : X \to Y$ *is open (almost open) iff for any open (regularly open) set* $A \subset X$, $F(A)$ *is open in Y,* |14|.

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THEOREM 1.1. *For a multifunction* $F : X \to Y$ *the following are equivalent:*

a) F *is u.s.c. (l.s.c.)* ,

b) $F^+(G)$ *is open (closed) for each open (closed) set* $G \subset Y$,

c) $F^-(G)$ *is closed (open) for each closed (open) set* $G \subset Y$, |1|.

THEOREM 1.2. *For a multifunction* $F : X \to Y$ *the following are equivalent:*

a) F *is u.a.c. (l.a.c.) ;*

b) $F^+(G)$ *is open (closed) for each regularly open (regularly closed) set* $G \subset Y$;

c) $F^-(G)$ *is closed (open) for each regularly closed (regularly open) set* $G \subset Y$, |15|.

DEFINITION 1.7. *A function* $f : X \to Y$ *is said to be almost continuous iff for each point* $x \in X$ *and each open neighbourhood* V *of* $f(x)$ *in* Y, *there exists an open neighbourhood* U *of* x *in* X *such that* $f(U) \subset \alpha(V)$, |16|.

A function is almost continuous iff the inverse image of every regularly open set is open, |16|.

DEFINITION 1.8. *A function* $f : X \to Y$ *is said to be almost closed (almost open) iff for every regularly closed (regularly open) set* F *of* X , $f(F)$ *is closed (open) in* Y, |16|.

2. SOME CHARACTERIZATIONS OF MULTIFUNCTIONS AND PARACOMPACTNESS

LEMMA 2.1. *If a multifunction* $F : X \to Y$ *is almost continuous and almost open, then*

a) *for each regularly open set* V *of* Y, $F^+(V)$ *is regularly open ;*

b) *for each regularly closed set* B *of* Y, $F^-(B)$ *is regularly closed.*

P r o o f. a) Let V be any regularly open set of Y. Since F is almost continuous $F^+(V)$ is open, hence we have

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$F^+(V) \subset \alpha(F^+(V))$. On the other hand, since F is almost continuous and $\bar{V}$ is regularly closed, $F^+(\bar{V})$ is closed, hence we have $\alpha(F^+(V)) \subset F^+(V) \subset F^+(\bar{V})$. Since $\alpha(F^+(V))$ is regularly open and F is almost open, then $F(\alpha(F^+(V)))$ is open. Hence we have $F(\alpha(F^+(V))) \subset F(F^+(\bar{V})) \subset \bar{V}$, i.e. $F(\alpha(F^+(V))) \subset \alpha(V) = V$. Hence we have $\alpha(F^+(V)) \subset F^+(V)$. Hence $\alpha(F^+(V)) = F^+(V)$, i.e. $F^+(V)$ is regularly open.

b) Let B be any regularly closed subset of Y. Then $Y \setminus B$ is regularly open and $F^-(B) = F^-(Y \setminus (Y \setminus B)) = X \setminus F^+(Y \setminus B)$. Since $F^+(Y \setminus B)$ is regularly open, then $F^-(B)$ is regularly closed.

COROLLARY 2.1. (|10|) *If a mapping* $f : X \to Y$ *is almost continuous and almost open, then*

a) *for each regularly open set* V *of* Y, $f^{-1}(V)$ *is regularly open,*

b) *for each regularly closed set* B *of* Y, $f^{-1}(B)$ *is regularly closed.*

LEMMA 2.2. *A surjective multifunction* $F : X \to Y$ *is almost closed iff for every subset* $S \subset Y$ *and any regularly open set* U *of a space* X *containing* $F^-(S)$, *there exists an open set* V *in* Y *such that* $S \subset V$ *and* $F^-(V) \subset U$.

P r o o f. Let F be any almost closed multifunction of a space X onto a space Y. Let S be any set of Y and U be any regularly open set in X containing $F^-(S)$. Let $V = Y \setminus F(X \setminus U)$. Then, since $F^-(S) \subset U$, we have $S \subset V$. Since U is regularly open and F is almost closed, then V is open in Y. Now we have $F^-(V) = X \setminus F^+(F(X \setminus U)) \subset U$.

Now, let A be any regularly closed subset of X and $y \in Y \setminus F(A)$ be any point. Then we have $F^-(y) \subset F^-(Y \setminus F(A)) = X \setminus F^+(F(A)) \subset X \setminus A$. There exists an open set V in Y such that $y \in V$ and $F^-(V) \subset X \setminus A$. Thus we have $y \in V \subset Y \setminus F(A)$, hence $Y \setminus F(A)$ is open. Therefore, F(A) is closed, i.e. F is almost closed.

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COROLLARLY 2.2. *(|10|) A surjective mapping* $f : X \to Y$ *is almost closed iff for any subset* $S \subset Y$ *and any regularly open set* U *of* X *containing* $f^{-1}(S)$, *there exists an open set* V *in* Y *such that* $S \subset V$ *and* $f^{-1}(V) \subset U$.

THEOREM 2.1. *If* X *is regular and multifunction* $F : X \to Y$ *is an almost closed surjection such that* $F^{-}(y)$ *is* α*-paracompact for each point* $y \in Y$, *then* F *is closed.*

P r o o f. Let A be any closed subset of X and let $y \in Y \setminus F(A)$. Since $F^{-}(y) \subset X \setminus A$ and X is regular, then for each point $x \in F^{-}(y)$ there exist open sets U_x and V_x such that $x \in U_x$, $A \subset V_x$, $U_x \cap V_x = \emptyset$. Now, $\mathcal{U} = \{U_x : x \in F^{-}(y)\}$ is an open covering of $F^{-}(y)$. Since $F^{-}(y)$ is α-paracompact, there exists an open X-locally finite family $\mathcal{W}$ which refines $\mathcal{U}$ and covers $F^{-}(y)$. For each $W \in \mathcal{W}$, there exists $U_{x_W} \in \mathcal{U}$ such that $W \subset U_{x_W}$. Since $U_{x_W} \cap V_{x_W} = \emptyset$, we have $\bar{W} \cap V_{x_W} = \emptyset$, i.e. $\bar{W} \cap A = \emptyset$. Let $V = \mathbf{U}\{W : W \in \mathcal{W}\}$. Since $\bar{V} = \overline{\mathbf{U}\{W : W \in \mathcal{W}\}} = \mathbf{U}\{\bar{W} : W \in \mathcal{W}\}$ we have $\bar{V} \cap A = \emptyset$. Since $\overline{\alpha(V)} = \bar{V}$, we have $\overline{\alpha(V)} \cap A = \emptyset$. Therefore $\alpha(V)$ is a regularly open set containing $F^{-}(y)$ such that $F^{-}(y) \subset \alpha(V) \subset X \setminus A$. Since F is almost closed, then there exists an open set G in Y such that $y \in G$ and $F^{-}(G) \subset \alpha(V) \subset X \setminus A$. Therefore we have, $y \in G \subset Y \setminus F(A)$. Hence $Y \setminus F(A)$ is open i.e. $F(A)$ is closed. Hence F is closed.

COROLLARY 2.3. *(|6|) If* X *is regular and* $f : X \to Y$ *is an almost closed surjection such that* $f^{-1}(y)$ *is* α*-paracompact for each point* $y \in Y$, *then* f *is closed.*

LEMMA 2.3. *Let a multifunction* $F : X \to Y$ *be an almost closed surjection such that* $F^{-}(y)$ *is* α*-nearly compact for each point* $y \in Y$. *If* $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ *is an open locally finite family then* $F(\mathcal{U}) = \{F(U_\alpha) : \alpha \in I\}$ *is a locally finite family.*

P r o o f. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any open locally finite family in a space X. Let $y \in Y$ be any point in Y. Then for each point $x \in F^{-}(y)$ there exists an open neighbourhood $G(x)$ of

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x such that $G(x)$ interesects finitely many members of $\mathcal{U}$. Let $I(x)$ be a finite subset of I such that $G(x) \cap U_\alpha \neq \emptyset$, $\alpha \in I(x)$ and $G(x) \cap U_\alpha = \emptyset$, $\alpha \in I \setminus I(x)$.

The family $\{G(x) : x \in F^-(y)\}$ is an X-open cover of $F^-(y)$. Since $F^-(y)$ is α-nearly compact, there exists a finite number of points $x_1, x_2, \ldots, x_n$ in $F^-(y)$ such that $F^-(y) \subset \mathbf{U}\{\alpha(G(x_i)) : i=1,2,\ldots,n\}$. Now, let $G = (\mathbf{U}\{\overline{(G(x_i)}: i=1,2,\ldots,n\})^o$. G is a regularly open set of X containing $F^-(y)$ such that $G \cap U_\alpha = \emptyset$, $\alpha \in I \setminus \bigcup_{i=1}^{n} I(x_i)$. If $G = X$, then a family $\mathcal{U}$ is finite, hence $F(\mathcal{U})$ is finite, i.e. locally finite. Let $G \neq X$. Since F is almost closed, then there exists an open set V of Y containing y such that $F^-(V) \subset G$. Thus we have $V \cap F(U_\alpha) = \emptyset$, for every $\alpha \in I \setminus \bigcup_{i=1}^{n} I(x_i)$. This implies that $F(\mathcal{U}) = \{F(U_\alpha) : \alpha \in I\}$ is locally finite.

COROLLARY 2.4. *(|11|) Let $f : X \to Y$ be an almost closed surjection with N-closed point inverses. If $\{U_\alpha : \alpha \in I\}$ is a locally finite open cover of X, then $\{f(U_\alpha) : \alpha \in I\}$ is a locally finite cover of Y.*

THEOREM 2.2. *If F is an almost closed u.s.c. and an open multifunction of a paracompact space X onto a space Y such that $F^-(y)$ is α-nearly compact for each point $y \in Y$ and $F(x)$ is α-paracompact for each point $x \in X$, then Y is paracompact.*

P r o o f. Let $\mathcal{U} = \{U_\alpha : \alpha \in I\}$ be any open covering of Y. Since for each point $x \in X$ the set $F(x)$ is α-paracompact and $\mathcal{U}$ is Y-open covering of $F(x)$, then there exists a Y-open Y-locally family $\{G_\lambda : \lambda \in I_x\}$ which refines $\mathcal{U}$ and is such that $F(x) \subset \mathbf{U}\{G_\lambda : \lambda \in I_x\}$. Let $G_x = \mathbf{U}\{G_\lambda : \lambda \in I_x\}$. G_x is an open set such that $F(x) \subset G_x$. Since F is u.s.c. then $\{F^+(G_x) : x \in X\}$ is an open cover of X. Since X is paracompact, there exists an open locally finite refinement $\mathcal{V} = \{V_\beta : \beta \in J\}$ of $\{F^+(G_x) : x \in X\}$. Since F is almost closed and $F^-(y)$ is α-nearly compact for each point $y \in Y$, then $F(\mathcal{V}) = \{F(V_\beta) : \beta \in J\}$ is locally finite. Since F is open and $\mathcal{V}$ is an open locally finite covering of X it is obvious that $F(\mathcal{V})$ is an open locally finite covering of Y. For

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each $\beta \in J$ there exists an element $x_\beta \in X$ such that $V_\beta \subset F^+(G_{x_\beta})$, i.e. $F(V_\beta) \subset F(F^+(G_{x_\beta})) \subset G_{x_\beta}$. Let

$$V^* = \{F(V_\beta) \cap G_\lambda : \lambda \in I_{x_\beta},\ \beta \in J\} .$$

V^* is an open locally finite refinement of U, hence Y is paracompact.

COROLLARY 2.5. (|13|) *If F is a closed u.s.c. and an open multifunction of a paracompact space X onto a space Y such that $F^-(y)$ is compact for each point $y \in Y$ and $F(x)$ is compact for each point $x \in X$, then Y is paracompact.*

COROLLARY 2.6. (|5|) *If $f : X \to Y$ is an almost closed continuous and open mapping of a paracompact space X onto a space Y such that $f^{-1}(y)$ is α-nearly compact for each point $y \in Y$, then Y is paracompact.*

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REZIME

O VIŠEZNAČNIM PRESLIKAVANJIMA I PARAKOMPAKTNOSTI

U radu se ispituju neke osobine odozgo skoro neprekidnih, odozdo skoro neprekidnih i skoro neprekidnih višeznačnih preslikavanja kao i kako se preslikavaju parakompaktni prostori pri višeznačnim preslikavanjima koja imaju i neke dodatne osobine.

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ON WONG'S CONJUCTURE

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ABSTRACT

In this paper we prove the existence of the differential equation

$$y'' + a(x)y = 0, \quad x \geq x_1$$

such that all the solutions are in $L^2[x_1,\infty)$ but at the same time they are not bounded.

J.S.W.Wong |1| gave the following conjucture: "If all solutions of the equation

(*) $\quad y'' + a(x)y = 0, \quad x \geq a$

belong to $L^2[a,\infty)$ then it is Lagrange stable too".

The aim of this paper is to give a counterexample.

Because of the theorem, given by Wong and Patula |2|, which states that if (*) is nonoscillatory then it is a limit point too, our solutions have to be oscillatory functions.

Let

$$y_1(x) = e^{-\frac{x}{2}} \sin e^x + \bar{\delta}(x), \quad x \geq x_1 ,$$

where

$$\delta(x) = \frac{(x-x_n)^3 (x_{n+1}-x)^3}{(x_{n+1}-x_n)^6} \log_2 n \operatorname{sgn}(\sin e^x), \quad x \in [x_n, x_{n+1}) \equiv J_n$$

$$\bar{\delta}(x) = \begin{cases} 0 , & x \notin J_{2^n} \\ \delta(x), & x \in J_{2^n} \end{cases}$$

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and

$$x_n = \ln\ln \pi, \quad n \in N.$$

As $\max_{x \in J_n} |\delta(x)| = 2^{-6}\log_2 n$ it follows that $y_1(x)$ is unbounded for $x \in [x_1, \infty)$. The following calculation

$$\int_{x_1}^{\infty} \bar{\delta}^2(x)dx = \sum_{n=o}^{\infty} \int_{J_{2^n}} \delta^2(x)dx = \sum_{n=o}^{\infty} 2^{-12} n^2 \ln(1 + 2^{-n}) \leq$$

$$\leq 2^{-12} \sum_{n=o}^{\infty} \frac{n^2}{2^n} < \infty$$

shows that $y_1(x) \in L^2[x_1, \infty)$.

For equation (*), the second linearly independent solution is given by

$$y^2(x) = y_1(x) \int^x \frac{dt}{y_1^2(t)}$$

which implies that

$$y_2^2(x) = y_1^2(x)\Big(\int^x \frac{dt}{y_1^2(t)}\Big)^2 \leq y_1^2(x) \int^x \frac{e^t dt}{\sin^2 e^t} = y_1^2(x)\operatorname{ctg}^2 e^x \leq$$

$$\leq 2(e^{-x}\sin^2 e^x + \bar{\delta}^2(x))\operatorname{ctg}^2 e^x = 2e^{-x}\cos^2 e^x + 2\bar{\delta}^2(x)\operatorname{ctg}^2 e^x.$$

Obviously $e^{-x}\cos^2 e^x \in L[x_1, \infty)$ and it remains to be proved that $\bar{\delta}^2(x)\operatorname{ctg}^2 e^x \in L[x_1, \infty)$.

Denote by x'_{2^n} and x''_{2^n}, $x'_{2^n} < x''_{2^n}$, the solutions of the equation $\operatorname{ctg}^2 e^x = 1$, $x \in J_{2^n}$. Then

$$\int_{x_1}^{\infty} \bar{\delta}^2(x)\operatorname{ctg}^2 e^x dx = \sum_{n=o}^{\infty} \int_{J_{2^n}} \bar{\delta}^2(x)\operatorname{ctg}^2 e^x dx =$$

$$(**) \quad = \sum_{n=o}^{\infty} \int_{x_{2^n}}^{x'_{2^n}} \delta^2(x)\operatorname{ctg}^2 e^x dx + \sum_{n=o}^{\infty} \int_{x'_{2^n}}^{x''_{2^n}} \delta^2(x)\operatorname{ctg}^2 e^x dx +$$

$$+ \sum_{n=o}^{\infty} \int_{x''_{2^n}}^{x_{2^n+1}} \delta^2(x)\operatorname{ctg}^2 e^x dx .$$

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We shall prove that all the three series of (**) are convergent. It is obvious for the second one. For the first we have

$$\sum_{n=0}^{\infty} \int_{x_{2^n}}^{x'_{2^n}} \delta^2(x)\operatorname{ctg}^2 e^x dx \leq \sum_{n=0}^{\infty} \int_{x_{2^n}}^{x'_{2^n}} (x-x_{2^n})^6 (x_{2^n+1}-x)^6 n^2 (x_{2^n+1} -$$

$$- x_{2^n})^{-12} \operatorname{ctg}^2 e^x dx \leq \sum_{n=0}^{\infty} (x_{2^n+1} - x_{2^n})^{-2} n^2 \int_{x_{2^n}}^{x'_{2^n}} \frac{(x-x_{2^n})^2}{\sin^2 e^x} dx < \sum_{n=0}^{\infty} \frac{n^2}{2^n} < \infty$$

because of

$$\frac{1}{2^n \pi} \leq \left| \frac{x-x_{2^n}}{\sin e^x} \right| \leq (x'_{2^n} - x_{2^n})\sqrt{2} < (x_{2^n+1} - x_{2^n}).$$

In a similar way we can prove that the third series is also convergent.

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REZIME

O JEDNOJ WONGOVOJ PRETPOSTAVCI

U radu je dokazana egzistencija diferencijalne jednačine

$$y'' + a(x)y = 0, \quad x \geq x_1$$

čija su sva rešenja iz $L^2[x_1,\infty)$ ali nisu i ograničena. Time je oborena pretpostavka J.S.W.Wonga.

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ASYMPTOTIC ANALYSIS OF A NONLINEAR SECOND ORDER DIFFERENCE EQUATION II

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ABSTRACT

The paper deals with the oscillatory and asymptotic behaviour of solutions of the nonlinear second order difference equation and gives the characterization of oscillation and its extreme solutions. Also, some specific results for the Emden-Fowler equation and some stability results for the linear equation are given.

1. INTRODUCTION

This paper deals with the oscillatory and asymptotic behaviour of solutions of the second order nonlinear difference equation of the form

$$(E) \quad \Delta(r(n)\Delta y(n)) + p(n+1)f(y(n+1)) = 0, \quad n=0,1,\dots,$$

where $\{r(n)\}_0^\infty$ and $\{p(n)\}_0^\infty$ are the given real sequences and $r(n) > 0$ for $n=0,1,\dots$, and Δ is the forward difference operator defined by $\Delta y(n) = y(n+1)-y(n)$.

Moreover, the function f is considered subject to condition

$$(O) \quad f \text{ is nondecreasing and } uf(u) > 0 \quad \text{for } u \neq 0.$$

It is known (see [2]) that the general linear difference operator of the second order can be presented as the first term of (E).

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By a solution of (E) we mean a real sequence $\{y(n)\}_0^\infty$ satisfying (E). Obviously, the solution $\{y(n)\}_0^\infty$ of (E) is uniquely determined by the initial values $y(0)$ and $y(1)$ or equivalently by any two successive values $y(k)$ and $y(k+1)$ and can be defined for all $n=0,1,\ldots$.

The following notions will be used in the sequel.

A real sequence $\{g(n)\}_0^\infty$ eventually has some property if there exists $N \geq 0$ such that $g(n)$ has this property for $n = N, N+1, \ldots$. Throughout this paper, we shall usually refer to a solution $\{y(n)\}_0^\infty$ of (E) simply as a solution y and considered only the nontrivial solutions y. A nontrivial solution y of (E) is said to be oscillatory if $y(n)$ changes the sign infinitely many times. Otherwise, y is said to be nonoscillatory.

Equation (E) is called oscillatory if each of its solution is oscillatory. Otherwise, it is called nonoscillatory.

The detailed discussion on various problems of the qualitative theory of the difference equation, including the oscillation and stability problems, can be found in [1] and [8]. We also cite papers [2], [3] and [4] for some further study on the problems of oscillation and asymptotic behaviour of solutions of (E).

Section 2. contains a fixed point theorem according to Knaster and a useful estimate for nonoscillatory solutions of (E). In Section 3., assuming that $p(n) \geq 0$ for $n \geq 0$, we obtain the characterization of some extreme types of nonoscillatory solutions. In Section 4. we give some oscillation results for (E), under the condition $\sum_0^\infty (r(n))^{-1} < \infty$ and we obtain the characterization of oscillation of (E).

Section 5. contains some specific results for the Emden-Fowler difference equation concerning the asymptotic behaviour of nonoscillatory solutions as well as some stability results for the linear equation (E) i.e. for the case where $f(u) = u$. The last section contains some extensions of the presented results.

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The results of this paper are a continuation of our previous results obtained in [7] which hold in the case $\sum_{0}^{\infty} (r(n))^{-1} = \infty$.

Throughout this paper, the phrase without loss of generality is abbreviated as WLOG.

2. PRELIMINARIES

In what follows, we shall use the following simple fixed point theorem according to Knaster.

THEOREM 2.1. *([9]) Let $\leq$ be a partial ordering with field A, and suppose that every $B \subseteq A$ has a least upper bound. Suppose that F maps A into A in such a way that for all x,y in A $x \leq y$ implies that $Fx \leq Fy$. Then $Fx = x$ for some $x \in A$.*

The following result gives useful information about the bounds for nonoscillatory solution of (E).

LEMMA 2.1. *Consider* (E) *subject to the conditions*

(C_1) $p(n) \geq 0$ *for* $n=0,1,\ldots$ *and* $p(n)$ *is not eventually zero, and*

(C_2) $$\sum_{n=0}^{\infty} \frac{1}{r(n)} < \infty$$

Then, every nonoscillatory solution y *of* (E) *satisfies eventually the following estimate*

(1) $$a\rho(n) \leq |y(n)| \leq A,$$

for some positive constants a and A *(depending on* y*) where*

$$\rho(n) = \sum_{i=n}^{\infty} \frac{1}{r(i)}.$$

P r o o f. WLOG we suppose that $y > 0$ eventually, which implies that $\Delta(r(n)\Delta y(n)) \leq 0$ eventually, which gives that $\Delta y(n)$ is eventually of constant sign.

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First, we suppose that $\Delta y(n) \geq 0$ eventually, i.e. $\Delta y(n) \geq 0$ for $n \geq N \geq 0$. Then, obviously $y(n) \geq y(N)$ for $n \geq N$ and moreover for $n \geq N$

$$0 \leq r(n)\Delta y(n) \leq C \Rightarrow y(n) \leq y(N) + C \sum_{i=N}^{n-1} \frac{1}{r(i)} \leq$$

$$\leq y(N) + C \sum_{i=o}^{\infty} \frac{1}{r(i)} = C_1 .$$

Secondly, we assume that $\Delta y(n) < 0$ eventually, i.e. $\Delta y(n) \leq 0$ for $n \geq N \geq 0$. Obviously, we may suppose that $y(n) > 0$ for $n \geq N$. Thus, we get $y(n) \leq y(N)$, $n \geq N$ and $r(n)\Delta y(n) \leq -C$ for some constant $C > 0$ and $n \geq N$. Now, we have for $n \geq N$

$$y(k) - y(n) \leq -C \sum_{i=n}^{k-1} \frac{1}{r(i)} \Rightarrow y(n) \geq y(k) + C \sum_{i=n}^{k-1} \frac{1}{r(i)} \geq$$

$$\geq y(\infty) + C \sum_{i=n}^{k-1} \frac{1}{r(i)} ,$$

which by $k \to \infty$, yields $y(n) \geq C_1 > 0$ if $y(\infty) > 0$ and $y(n) \geq a\rho(n)$ if $y(\infty) = 0$.

Taking into account Lemma 2.1., it is natural to introduce the following terminology: a (nonoscillatory) solution asymptotically equivalent to $a\rho(n)$ $(n \to \infty)$ will be called the solution of the minimal type, while the solution which is asymptotic to a nonzero constant will be called the solution of the maximal type.

3. CHARACTERIZATION OF THE MINIMAL AND MAXIMAL TYPE SOLUTIONS

In this section we shall give the necessary and sufficent conditions which provide the existence of the minimal and maximal types solutions.

THEOREM 3. 1. *Consider the equation* (E) *subject to conditions* (C_1) *and* (C_2). *Then* (E) *has a nonoscillatory solution of the maximal type iff the following condition holds*

$$(C_3) \qquad \sum_{n=o}^{\infty} \rho(n)p(n) < \infty .$$

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P r o o f. First, we suppose that (C_3) holds. Then, there exist $N \geq 0$ such that

$$f(2K) \sum_{n=N}^{\infty} \rho(n)p(n) \leq K ,$$

where $K > 0$ is some arbitrary constant.

Let S be the set of all nonincreasing sequences x with the property

$$K \leq x(n) \leq 2N \quad \text{for} \quad n \geq N .$$

The set S is considered endowed with the usual point-wise ordering $\leq$:

$$x_1 \leq x_2 \iff (\forall n \geq N)\ x_1(n) \leq x_2(n) .$$

Moreover, we consider the mapping F on S defined as follows:

$$(Fx)(n) = K + \rho(n) \sum_{i=N}^{n-1} p(i+1)f(x(i+1)) +$$

$$+ \sum_{i=n}^{\infty} \rho(i+1)p(i+1)f(x(i+1)), \quad n = N, N+1, \ldots .$$

Obviously, all conditions of Theorem 2.1. are satisfied, which implies that there exists $w \in S$ such that $Fw = w$, i.e.

$$w(n) = K + \rho(n) \sum_{i=N}^{n-1} p(i+1)f(x(i+1)) +$$

$$+ \sum_{i=n}^{\infty} \rho(i+1)p(i+1)f(x(i+1)), \quad n \geq N.$$

Now, it is easy to see that w is a required nonoscillatory solution of (E). Next, we suppose that (E) has a nonoscillatory solution y such that $\lim_{n\to\infty} y(n) = A > 0$. Then, by Lemma 2.1., we conclude that $\Delta y(n)$ is eventually of constant sign. Hence, there exists $N \geq 0$ such that

$$n \geq N \Rightarrow y(n) \geq \frac{A}{2} \quad \text{and} \quad \Delta y(n) \geq 0 \quad \text{or} \quad \Delta y(n) \leq 0 .$$

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If $\Delta y(n) \geq 0$, then we introduce the sequence $U(n) = \rho(n) r(n) \Delta y(n)$ and get

$$\Delta U(n) = \rho(n+1)\Delta(r(n)\Delta y(n)) - \Delta y(n) =$$

$$= -\rho(n+1)p(n+1)f(y(n+1)) - \Delta y(n) ,$$

which by summation from N to k-1, $k > N$, yields

$$0 \leq U(k) \leq U(N) + y(N) - y(k) - \sum_{n=N}^{k-1} \rho(n+1)p(n+1)f(y(n+1)) \leq$$

$$\leq C - \sum_{n=N}^{k-1} \rho(n+1)p(n+1)f(y(n+1)) ,$$

where $C = U(N) + \frac{A}{2}$. Thus, we obtain

$$f(\tfrac{A}{2}) \sum_{n=N}^{\infty} \rho(n+1)p(n+1) \leq \sum_{n=N}^{\infty} \rho(n+1)p(n+1)f(y(n+1)) \leq C .$$

In the case where $\Delta y(n) \leq 0$ we again obtain the relation

$$U(k) \leq C - \sum_{n=N}^{k-1} \rho(n+1)p(n+1)f(y(n+1)), \quad k > N,$$

which implies

$$U(k) \leq C - f(A) \sum_{n=N}^{k-1} \rho(n+1)p(n+1), \quad k > N.$$

If (C_3) fails, the last relation gives $U(k) \leq -1$ for $k \geq M \geq N$, which immediately implies

$$\Delta y(k) \leq - \frac{1}{r(k)\rho(k)} , \quad k \geq M ,$$

and summing from M to $n > M$ we have

$$y(n) \leq y(M) - \sum_{k=M}^{n-1} \frac{1}{r(k)\rho(k)} = y(M) +$$

$$+ \sum_{k=M}^{n-1} \frac{\Delta\rho(k)}{\rho(k)} \sim y(M) + \ln \rho(n-1) \quad \text{as} \quad n \to \infty .$$

So, $\lim_{n\to\infty} y(n) = -\infty$, which is an immediate contradiction.

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THEOREM 3.2. *Consider the difference equation* (E) *subject to the conditions* (C_1) *and* (C_2). *Then* (E) *has a nonoscillatory solution of the minimal type iff*

(C_4) $\sum_{n=0}^{\infty} p(n)|f(\alpha\rho(n))| < \infty$, *for some constant* $\alpha \neq 0$, *holds.*

P r o o f. First, we assume that (C_4) holds. Then, there exists $N \geq 0$ such that

$$\sum_{n=N} p(n) f(\alpha\rho(n)) \leq \frac{\alpha}{2} .$$

Let S be the set of all nonincreasing sequences x such that

$$\frac{\alpha}{2}\rho(n) \leq x(n) \leq \alpha\rho(n), \quad n \geq N .$$

S is considered endowed with the usual point-wise ordering $\leq$ as in the proof of Theorem 3.1. Moreover, we consider the mapping G on S defined in the following way:

$$(Gx)(n) = \frac{\alpha}{2}\rho(n) + \sum_{i=n}^{\infty} \frac{1}{r(i)} \sum_{j=N}^{i-1} p(j+1) f(x(j+1)), \quad n=N+1,\ldots .$$

Obviously, all conditions of Theorem 2.1. are satisfied, which implies that G has a fixed point z i.e. $Gz = z$ and it is clear that z is required solution of (E).

Next, suppose that y is a nonoscillatory solution of the minimal type. Taking into account Lemma 2.1. we conclude that y is a nonincreasing sequence for which there exists $M \geq 0$ such that

$$n \geq M \Longrightarrow y(n) \geq \alpha\rho(n) \quad \text{and} \quad \Delta y(n) \leq 0 \quad \text{where } \alpha > 0 \text{ is constant.}$$

Now, (E) implies

$$r(n)\Delta y(n) + \sum_{i=N}^{n-1} p(i+1) f(y(i+1)) = r(N)\Delta y(N) \quad \text{for} \quad n > N.$$

On the other hand, in [7] the following identity was obtained

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$$(2)\quad y(k+1) = y(N) - r(M)\Delta y(M) \sum_{i=k}^{N} \frac{1}{r(i)} +$$

$$+ \Big(\sum_{i=k}^{N} \frac{1}{r(i)} \sum_{i=M-1}^{k} p(i+1)f(y(i+1)) +$$

$$+ \sum_{n=k-1}^{N} \Big(\sum_{i=n}^{N} \frac{1}{r(i)} \Big) p(n+1)f(y(n+1)), \quad N > k > M,$$

which immediately implies

$$y(k+1) \geq \Big(\sum_{i=k}^{N} \frac{1}{r(i)} \Big) \sum_{i=M-1}^{k} p(i+1)f(y(i+1)), \quad N > k > M$$

and so

$$(3)\quad y(k+1) \geq \rho(k) \sum_{i=M-1}^{k} p(i+1)f(y(i+1)) \geq$$

$$\geq \rho(k+1) \sum_{i=M-1}^{k} p(i+1)f(y(i+1)) .$$

Hence, we get

$$\sum_{i=M}^{\infty} p(i)f(\alpha\rho(i)) \leq \sum_{i=M}^{\infty} p(i)f(y(i)) \leq \lim_{k\to\infty} \frac{y(k)}{\rho(k)} < \infty ,$$

which completes the proof of the theorem.

4. OSCILLATION THEOREMS

In this section we shall present some oscillation results for (E) which are the complements to the results of the preceding section.

THEOREM 4.1. *Consider the difference equation* (E) *subject to condition* (C_1), (C_2),

$$(C_5)\qquad \sum_{n=o}^{\infty} \rho(n)p(n) = \infty ,$$

and

$$(C_6)\qquad \sum_{m=o}^{\infty} \frac{1}{\rho(m)r(m)} \sum_{i=o}^{m} \rho(i)p(i) = \infty .$$

If the following condition holds

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(C_7) $$\int_0^{\pm\delta} \frac{du}{f(u)} < \infty \ , \quad \delta > 0,$$

then (E) *is oscillatory.*

P r o o f. Let x be a nonoscillatory solution of (E). Then, x is of constant sign eventually, say for $n \geq N$. Let

$$w(n) = \frac{r(n)\Delta x(n)}{f(x(n))}\,\rho(n), \quad n \geq N,$$

then we have

$$\Delta w(n) = \rho(n+1)\Delta\Big(\frac{r(n)\Delta x(n)}{f(x(n))}\Big) + \frac{r(n)\Delta x(n)}{f(x(n))}\,\Delta\rho(n) =$$

$$= \rho(n+1)\Big[\frac{\Delta r(n)\Delta x(n))}{f(x(n+1))} - \frac{r(n)\Delta x(n)\Delta f(x(n))}{f(x(n))f(x(n+1))}\Big] - \frac{\Delta x(n)}{f(x(n))} =$$

$$= -\rho(n+1)p(n+1) - \rho(n+1)\,\frac{r(n)x(n)\Delta f(x(n))}{f(x(n))f(x(n+1))} - \frac{\Delta x(n)}{f(x(n))},$$

which implies

$$\Delta w(n) \leq -\rho(n+1)p(n+1) - \frac{\Delta x(n)}{f(x(n))}, \quad n > N,$$

and by summation from N to m-1, $m > N$, we get

$$(4) \qquad w(m) \leq w(N) - \sum_{n=N}^{m-1} \rho(n+1)p(n+1) - \sum_{n=N}^{m-1} \frac{\Delta x(n)}{f(x(n))}, \quad m > N.$$

Now, taking into account conditions (C_5) and (C_7), we conclude that $\lim_{m\to\infty} w(m) = -\infty$ and consequently there exists $M > N$ such that $x(n)\Delta x(n) \leq 0$ for every $n \geq M$ and so $\lim_{n\to\infty} |x(n)| = \delta$, $\delta \geq 0$.

Hence, using (C_5) and (C_7), (4) implies

$$w(m) \leq -\frac{1}{2} \sum_{n=\tilde{N}+1}^{m} \rho(n)p(n) \quad \text{for} \quad m \geq \tilde{N}$$

where $\tilde{N}$ is chosen appropriately. Thus, we have

$$\frac{\Delta x(m)}{f(x(m))} \leq -\frac{1}{2}\,\frac{1}{\rho(m)r(m)} \sum_{n=\tilde{N}+1}^{m} \rho(n)p(n), \quad m \geq \tilde{N},$$

which gives

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$$\sum_{m=\tilde{N}}^{k} \frac{\Delta x(m)}{f(x(m))} \leq -\frac{1}{2} \sum_{m=\tilde{N}}^{k} \frac{1}{\rho(m)r(m)} \sum_{n=N+1}^{m} \rho(n)p(n), \quad k > \tilde{N}.$$

Taking $k \to \infty$ and using conditions (C_6) and (C_7) we obtain an immediate contradiction, which completes the proof the theorem.

THEOREM 4.2. *Consider the difference equation* (E) *subject to conditions* (C_1), (C_2) *and*

$$(C_8) \qquad \sum_{n=o}^{\infty} \rho(n)p(n)|f(\alpha\rho(n))| = \infty, \quad \textit{for all} \quad \alpha \neq 0.$$

Then (E) *is oscillatory.*

P r o o f. Let y be a nonoscillatory solution of (E). Then, y is of constant sign eventually and estimate (1) holds, which means that there exists $N \geq 0$ such that

$$n \geq N \Rightarrow a\rho(n) \leq |y(n)| \leq A \text{ and } \Delta y(n) \text{ is of a constant sign.}$$

WLOG we may suppose that $y > 0$. Thus, by (C_8), we conclude

$$(5) \qquad \sum_{n=o}^{\infty} \rho(n)p(n)f(y(n)) = \sum_{n=o}^{\infty} \rho(n)p(n) = \infty,$$

and summing (E) from N to n-1, we obtain

$$r(n)\Delta y(n) - r(N)\Delta y(N) + \sum_{i=N}^{n-1} p(i+1)f(y(i+1)) = 0,$$

which by (5) implies $\lim_{n\to\infty} r(n)\Delta y(n) = -\infty$. So, $\Delta y(n) \leq 0$ eventually. WLOG we may suppose that $\Delta y(n) \leq 0$ for $n \geq N$. Introducing the sequence $U(n)$ as in the proof of Theorem 3.1., we obtain the relation

$$U(k) + y(k) \leq C - \sum_{n=N}^{k-1} \rho(n+1)p(n+1)f(y(n+1)),$$

where $C = U(N) + y(N)$.

Now it is easy to show that $U(k) + y(k) \geq 0$ eventually, in which case the last inequality implies the required conclusion.

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Actually, multiplying (E) by $\sum_{i=n+1}^{k} \frac{1}{r(i)}$ and then summing from m-1 to k-1 we get

$$\sum_{n=m-1}^{k-1} \Big(\sum_{i=n+1}^{k} \frac{1}{r(i)} \Big) \Delta(r(n)\Delta y(n)) +$$

$$+ \sum_{n=m-1}^{k-1} \Big(\sum_{i=n+1}^{k} \frac{1}{r(i)} \Big) p(n+1) f(y(n+1)) = 0,$$

which by the summation by parts formula, implies

$$(6) \quad y(m) + \Big(\sum_{i=m}^{k} \frac{1}{r(i)} \Big) r(m)\Delta y(m) = y(k+1) +$$

$$+ \sum_{n=m-1}^{k-1} \Big(\sum_{i=n+1}^{k} \frac{1}{r(i)} \Big) p(n+1) f(y(n+1)) \geq 0 .$$

Thus, we obtain

$$y(m) + \rho(m) r(m) \Delta y(m) \geq 0,$$

which completes the proof of the theorem.

COROLLARY 4.1. *Consider the difference equation* (E) *subject to conditions* (C_1), (C_2) *and* (C_7). *Then,* (E) *is oscillatory iff condition* (C_5) *holds.*

P r o o f. It follows immediately from Theorem 3.1. and 4.1., taking into account the fact that (C_2) and (C_5) imply (C_5) and (C_6).

REMARK 4.1. Theorem 4.1. can be considered a partial discrete analogue of a result of Kulenović and Grammatikopoulos [5] pertaining to the corresponding differential equation. Furthermore, Theorem 4.2. is a dicsrete analogue of a result of Kulenović [6].

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5. APPLICATION TO THE EMDEN-FOWLER EQUATION

In this section we shall consider the Emden-Fowler difference equation

$$(EF)\quad \Delta(r(n)\Delta y(n)) + p(n+1)|y(n+1)|^{\nu}\operatorname{sgn} y(n+1) = 0,\ \nu > 0,$$

$n=0,1,\ldots,$.

and obtain some on the global asymptotic behaviour of its oscillatory and nonoscillatory solutions.

THEOREM 5.1. *Consider the difference equation* (EF) *subject to the conditions* (C_1) *and* (C_2). *If* (C_3) *holds and* $\nu \geq 1$, *then every nonoscillatory solution of* (EF) *is either of the minimal or of the maximal type.*

P r o o f. Since $\nu \geq 1$, condition (C_3) implies (C_4) and by Theorem 3.1. and 3.2. the equation (EF) has both types of solutions. Now, we shall prove that (EF) has no other types of solutions.

Let y be an arbitrary nonoscillatory solution of (EF). WLOG we can suppose that $y > 0$ eventually, which implies that $\Delta y(n)$ is eventually of constant sign. If $\Delta y(n) \geq 0$ eventually, we have the conclusion of the theorem. Otherwise $\Delta y(n) \leq 0$ eventually and we can suppose that $\lim_{n\to\infty} y(n) = 0$. Thus there exists $N \geq 0$ such that

$$n \geq N \Longrightarrow y(n) > 0 \text{ and } \Delta y(n) \leq 0 \text{ and the estimate (1) holds.}$$

Starting from relation (6) and taking $k \to \infty$, we obtain

$$y(m) = \sum_{n=m-1}^{\infty} \rho(n+1)p(n+1)(y(n+1))^{\nu} - \rho(m)r(m)\Delta y(m),\quad m \geq N,$$

which by estimate (1) and condition (C_3) yields

$$y(m) \leq y(m) \sum_{n=m}^{\infty} \rho(n)p(n)(y(n))^{\nu-1} - \rho(m)r(m)\Delta y(m) \leq$$

$$\leq A^{\nu-1} y(m) \sum_{n=N}^{\infty} \rho(n)p(n) - \rho(m)r(m)\Delta y(m) \leq$$

$$\leq \varepsilon y(m) - \rho(m)r(m)\Delta y(m),\quad m \geq N,$$

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if N is large enough to provide $A^{\nu-1} \sum_{n=N}^{\infty} \rho(n)p(n) \leq \varepsilon$, for an arbitrary $\varepsilon > 0$.

Thus, we obtain

$$-\rho(m)r(m)\Delta y(m) \geq (1-\varepsilon)y(m), \quad m \geq N.$$

Now, taking into account this relation, estimate (1) and (C_3). we get

$$(1-\varepsilon)\left[r(N)\Delta y(N) - r(n)\Delta y(n)\right] = (1-\varepsilon) \sum_{m=N}^{n-1} p(m+1)(y(m+1))^{\nu} \leq$$

$$\leq - \sum_{m=N}^{n-1} p(m+1)(y(m+1))^{\nu-1} \rho(m+1)r(m+1)\Delta y(m+1) \leq$$

$$\leq -A^{\nu-1} r(n)\Delta y(n) \sum_{m=N}^{n} \rho(m)p(m) \leq$$

$$\leq - A^{\nu-1} r(n)\Delta y(n) \sum_{m=N}^{\infty} \rho(m)p(m) \leq -\varepsilon r(n)\Delta y(n) .$$

Thus, we conclude

$$(1-\varepsilon)\left[r(N)\Delta y(N) - r(n)\Delta y(n)\right] \leq - \varepsilon r(n)\Delta y(n) ,$$

which implies

$$r(n)\Delta y(n) \geq \frac{1-\varepsilon}{1-2\varepsilon} r(N)\Delta y(N),$$

and so $\lim_{n\to\infty} r(n)\Delta y(n) = \delta \in (-\infty, 0]$, which by the discrete L´ Hospital rule gives

$$\lim_{n\to\infty} \frac{y(n)}{\rho(n)} = - \lim_{n\to\infty} r(n)\Delta y(n) = -\delta ,$$

which completes the proof of the theorem.

Similar arguments lead to the corresponding result for the sublinear ($\nu \leq 1$) difference equation (EF).

THEOREM 5.2. *Consider the difference equation* (EF) *subject to conditions* (C_1) *and* (C_2). *If* $\nu \leq 1$ *and condition* (C_4) *holds, then every nonoscillatory solution of* (EF) *is either of the minimal or of the maximal type.*

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Now, we shall discuss the solutions of intermediate types and we shall get some sharper estimates than (1).

THEOREM 5.3. *Consider the difference equation* (EF) *subject to conditions* (C_1) *and* (C_2). *If the following condition holds*

$$(C_9) \qquad \lim_{n\to\infty} \inf (\rho(n))^{\lambda} \sum_{k=o}^{n} p(k) > 0$$

then, for every nonoscillatory solution of (EF) *the following estimate holds*

$$a(\rho(n))^{\frac{1-\lambda}{1-\nu}} \leq |y(n)| \leq A, \quad \textit{eventually}, \quad 0 < \nu < \lambda \leq 1,$$

and, if the additional condition

$$(C_{10}) \qquad \lim_{n\to\infty} \sup (\rho(n))^{\lambda} \sum_{k=o}^{n} p(k) = \infty$$

holds, then the following estimate holds:

$$(7) \qquad a\rho(n) \leq |y(n)| \leq A(\rho(n))^{\frac{\lambda-1}{\nu-1}}, \quad \textit{eventually}, \quad 1 < \lambda \leq \nu,$$

for some appropriate constants a *and* A *depending on the solution* y .

P r o o f. Assume that y is a nonoscillatory solution of (EF). WLOG we suppose that $y > 0$ eventually. Then, it is easy to conclude that $\Delta y(n) \leq 0$ eventually.

Actually, if $\Delta y(n) \geq 0$, in the first case, then we have the required conclusion. In the second case, using (C_{10}) we have

$$\sum_{k=N}^{n} \rho(k)p(k) \geq \sum_{k=N}^{n} (\rho(k))^{\lambda} p(k) \geq (\rho(n))^{\lambda} \sum_{k=N}^{n} p(k) \to \infty ,$$

which by Theorem 3.1. implies that $\lim_{n\to\infty} y(n) = 0$ and so $\Delta y(n) \leq 0$ eventually.

Let $M \geq 0$ be such that

$$n \geq M \Longrightarrow y(n) > 0 \quad \text{and} \quad \Delta y(n) \leq 0.$$

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Starting from identity (2), we get inequality (3) which implies

$$y(k+1) \geq \rho(k+1) \sum_{i=M-1}^{k} p(i+1)(y(i+1))^{\nu} \geq$$

$$\geq \rho(k+1)(y(k+1))^{\nu} \sum_{i=M}^{k+1} p(i) ,$$

and by condition (C_9) we get

$$1 \geq \rho(k+1)(y(k+1))^{\nu-1} \sum_{i=M}^{k+1} p(i) =$$

$$= \frac{(y(k+1))^{\nu-1}}{(\rho(k+1))^{\lambda-1}} (\rho(k+1))^{\lambda} \sum_{i=M}^{k+1} p(k) \geq C \frac{(y(k+1))^{\nu-1}}{(\rho(k+1))^{\lambda-1}} ,$$

for some $C > 0$, which immediately implies the required estimate.

REMARK 5.1. Since condition (C_{10}) implies condition (C_5), the second part of Theorem 5.3. can be understood as follows: if there exists a nonoscillatory solution of (EF) then estimate (7) holds.

COROLLARY 5.1. *Consider the difference equation* (EF) *subject to conditions* (C_1) *and* (C_2). *If* $\nu > 1$, *then* (EF) *is oscillatory iff the following condition holds*

$$(C_{11}) \qquad \sum_{n=o}^{\infty} p(n)(\rho(n))^{\nu} = \infty .$$

P r o o f. The necessery part follows from Theorem 3.2., while the sufficient part can be obtained following similar steps as in the proofs of Theorems 4.1. and 4.2..

Now, we shall give the result concerning the global behaviour of oscillatory solutions of (E).

THEOREM 5.4. *Consider the difference equation* (E) *subject to conditions* (C_1), (C_2) *and*

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$$(C_{12}) \qquad \sum_{n=0}^{\infty} \frac{1}{r(n)} \sum_{k=n+1}^{\infty} p(k) < \infty \ .$$

If f *is weakly sublinear i.e.* $\lim_{|u|\to\infty} \sup \frac{f(u)}{u} < \infty$, *then all oscillatory solutions of* (E) *tend to zero.*

P r o o f. If our assertion is not true, then there exist sequences of natural numbers $\{t(n)\}_0^\infty$ and $\{\pi(n)\}_0^\infty$ and the oscillatory solution x, such that $\lim_{n\to\infty} t(n) = \lim_{n\to\infty} \pi(n) = \infty$, $t(n+1) > t(n) + 1$ and $\pi(n) \in (t(n-1), t(n))$ with the properties $x(t(n))x(\pi(n)) \leq 0$ and $|x(t(n))| > \delta > 0$. WLOG we can suppose that $x(\pi(n)) \leq 0$ and $x(t(n)) = \max\{x(m) : t(n-1) < m < t(m+1)\}$.

Now, summing (E) from k to $t(n) - 1$, we obtain

$$r(t(n))\Delta x(t(n)) = r(k)\Delta x(k) - \sum_{i=k}^{t(n)-1} p(i+1) f(x(i+1)),$$

$n = 1, 2, \ldots$.

Now, taking into account the fact that $\Delta x(t(n)) = x(t(n)+1) - x(t(n)) \leq 0$, the last relation implies

$$r(k)\Delta x(k) \leq \sum_{i=k}^{t(n)-1} p(i+1) f(x(i+1)), \quad n = 1, 2, \ldots, t(n-1) \leq k < t(n)$$

which yields

$$\Delta x(k) \leq \frac{1}{r(k)} \sum_{i=k}^{t(n)-1} p(i+1) f(x(i+1)), \quad n = 1, 2, \ldots, t(n-1) \leq k < t(n)$$

and summing from $\pi(n)$ to $t(n)-1$, we have

$$x(t(n)) \leq x(\pi(n)) + \sum_{k=\pi(n)}^{t(n)-1} \frac{1}{r(k)} \sum_{i=k}^{t(n)-1} p(i+1) f(x(i+1)) \leq$$

$$\leq f(x(t(n))) \sum_{k=\pi(n)}^{t(n)-1} \frac{1}{r(k)} \sum_{i=k}^{t(n)-1} p(i+1) \leq$$

$$\leq f(x(t(n))) \sum_{k=\pi(n)}^{\infty} \frac{1}{r(k)} \sum_{i=k+1}^{\infty} p(i) \ .$$

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Thus, we obtain

$$1 \leq \frac{f(x(t(n)))}{x(t(n))} \sum_{k=\pi(n)}^{\infty} \frac{1}{r(k)} \sum_{i=k+1}^{\infty} p(i) ,$$

which is, by (C_{12}) and the conditions imposed on the function f, an immediate contradiction.

REMARK 5.2. Obviously condition (C_{12}) is implied by condition

$$(C'_{12}) \qquad \sum_{n=o}^{\infty} \frac{1}{r(n)} < \infty, \quad \sum_{n=o}^{\infty} p(n) < \infty ,$$

as well as by condition

$$(C''_{12}) \qquad \sum_{n=o}^{\infty} \frac{1}{r(n)} = \infty , \quad \sum_{n=o}^{\infty} R(n)p(n) < \infty, \text{ where } R(n) = \sum_{i=o}^{n-1} \frac{1}{r(i)} .$$

Since (C'_{12}) implies conditions (C_3) and (C_4) we can get the following result.

COROLLARY 5.2. *Consider the difference equation* (EF) *subject to conditions* (C_1) *and* (C'_{12}) . *If* $\nu \leq 1$, *then every nonoscillatory solution is either of the minimal or of the maximal type and every oscillatory solution tends to zero. If* $\nu = 1$, *it means that all solutions of* (EF) *are stable.*

Also, in the light of Theorem 5.4., we can get the following result.

COROLLARY 5.3. *Consider the difference equation* (EF) *subject to conditions* (C_1) *and* (C''_{12}). *If* $\nu \leq 1$, *then every nonoscillatory solution is either asymptotic to nonzero constant or to* AR(n) $(A \neq 0)$ *and every oscillatory solution tends to zero.*

P r o o f. It follows immediately from Theorem 5.4. and [7].

REMARK 5.3. Some further results on stability and

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asymptotic stability of the linear equation (EF) ($\nu = 1$) can be obtained by comparing the conditions of our results with the previous results of Hooker and Patula [3], where some stability results are given. We note that the linear second order difference equation

$$c(n+1)y(n+2) + c(n)y(n) = b(n+1)y(n+1), \quad n=0,1,\dots ,$$

where $c(n) > 0$, $n=0,1,\dots$, can always be reduced to the equation (EF) with $\nu = 1$

$$\Delta(c(n)\Delta x(n)) + a(n+1)\, x(n+1) = 0, \quad n=0,1,\dots ,$$

where $a(n) = c(n) + c(n-1) - b(n)$, $n=1,2,\dots$ (see |2| and |3|). Using this fact and the oscillation and nonoscillation results from |2| and |3| we can give some further results on the stability and asymptotic stability of (EF).

6. CONCLUDING REMARKS

REMARK 6.1. In the case of the Thomas-Fermi difference equation (E) i.e. in the case where condition (C_1) is replaced by the following one

(C_1') $p(n) \leq 0$ for $n=0,1,\dots$ and $p(n)$ is not eventually zero.

Using similar methods as in the proofs of Theorem 3.1. and 3.2., we can prove, under some appropriate conditions, the existence of solutions asymptotically equivalent to constant and to $\rho(n)$. Thus, we can prove the following

THEOREM 6.1. *Consider the difference equation* (E) *subject to condition* (C_1') *and* (C_2). *Then,* (E) *has a nonoscillatory solution of the minimal type iff the following condition holds*

$$\sum_{n=0}^{\infty} \frac{1}{r(n)} \sum_{k=0}^{n} p(k) < \infty .$$

(E) *has a solution of the maximal type iff condition* (C_4) *holds.*

We note that in the case of the Thomas-Fermi equation the estimate related to (1) does not hold and so the terms

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minimal and maximal do not have the same sense as above.

Moreover, all the results of Section 2.-4. can be extended to the corresponding difference inequality

$$y(n)\left[\Delta(r(n)\Delta y(n)) + p(n+1)f(y(n+1))\right] \leq 0, \quad n=0,1,\dots,$$

while Theorem 6.1. also holds for the following difference inequality

$$y(n)\left[\Delta(r(n)\Delta y(n)) + p(n+1)f(y(n+1))\right] \geq 0, \quad n=0,1,\dots .$$

REMARK 6.2. All the obtained results can be extended to the difference equations of the from

$$\Delta r(n)\Delta y(n)) + p(n)f(y(n)) = 0, \quad n=0,1,\dots,$$

and

$$\Delta(r(n)\Delta y(n)) + p(n+2)f(y(n+2)) = 0, \quad n=0,1,\dots ,$$

which are discrete approximations of the same differential equation $(ry')' + pf(y) = 0$. In that case, the conditions are slightly different, while the methods are the same.

Finally, we note that the presented results of Section 2.-4. also hold if condition (0) on f is replaced by the condition that f is either sublinear ($\frac{f(u)}{u}$ is nondecreasing for $u < 0$ and nonincreasing for $u > 0$) or superlinear ($\frac{f(u)}{u}$ is nonincreasing for $u < 0$ and nondecreasing for $u > 0$).

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REZIME

ASIMPTOTSKA ANALIZA NELINEARNE DIFERENTNE JEDNAČINE DRUGOG REDA (II)

U radu se izučava oscilatornost i asimptotsko ponašanje rešenja diferentne jednačine drugog reda.

(E) $\Delta(r(n)\Delta y(n)) + p(n+1)f(y(n+1)) = 0,\ n=0,1,\dots$.

Posle uvodnih izlaganja, u Glavi 2 se daje korisna ocena za neoscilatorna rešenja od (E). U Glavi 3 dobijamo karakterizaciju minimalnog i maksimalnog neoscilatornog rešenja. U Glavi 4 dobijamo karakterizaciju oscilatornih rešenja. Glava 5 sadrži neke specifične rezultate za Emden-Fowlerovu jednačinu kao i neke rezultate vezane za stabilnost linearne jednačine. Poslednja glava sadrži neka uopštenja i proširenja prethodnih rezultata.

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AN ANALYTIC CHARACTERIZATION OF THE SPACE OF GENERALIZED FUNCTIONS WHICH HAVE A LAGUERRE EXPANSION

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ABSTRACT

The space LG' of generalized functions, whose elements have a Laguerre orthonormal expansion into a series, are investigated in |2| and |3|. In this paper we define the space LG by using a suitable family of seminorms. This implies some properties of the space LG' and the representation theorem for some elements from LG'. Also, by using a convolution in LG' and a Laplace transform we give expansions into a series of some important generalized functions from LG'.

I In monograph |4|, Zemanian investigated spaces of generalized functions whose elements have an orthonormal expansion into a series. These spaces are denoted by A'.

In |4| various examples of such spaces are given. These spaces correspond to various spaces $L^2(I)$, (I is an interval in R) and their orthonormal bases (ψ_n). The best known space of A'-type is the space S'. Among the examples, the space of the test functions LG_α and of the generalized functions LG'_α, $\alpha > -1$, are given and these correspond to the space $L^2(0,\infty)$ and the generalized Laguerre orthonormal bases (ℓ_n^α) of $L^2(0,\infty)$, $\alpha > -1$, where

$$\ell_n^\alpha(x) := \left(\frac{\Gamma(n+1)}{\Gamma(\alpha+n+1)}\right)^{1/2} x^{\alpha/2} e^{-x/2} L_n^\alpha(x), \quad n \in N_o \ (= N \cup \{0\}).$$

Laguerre polynomials are given by

$$L_n^\alpha(x) := \sum_{m=0}^{n} \binom{n+\alpha}{n-m} \frac{(-x)^m}{m!} \qquad \left(\binom{y}{\nu} = \frac{\Gamma(y+1)}{\Gamma(\nu-1)\Gamma(y-\nu-1)}\right).$$

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For a fixed $\alpha > -1$, the functions $\ell_n^\alpha(x)$, $n \in N_o$, are the eigenfunctions for the self-adjoint operator in $L^2(0,\infty)$

$$R_\alpha := x^{-\alpha/2} e^{x/2} D x^{\alpha+1} e^{-x} D x^{-\alpha/2} e^{x/2} \quad (D = \frac{d}{dx})$$

for which

$$R_\alpha \ell_n^\alpha = -n\ell_n^\alpha , \quad n = 0,1,... \tag{1}$$

hold.

In |2| and |3| we investigated the spaces LG and LG′ which correspond to $\alpha = 0$, $LG := LG_o$, $LG' = LG_o'$, because the base (ℓ_n), $\ell_n = \ell_n^o$, $n \in N_o$, gives some conveniences. For example in LG′ the convolution and Laplace transform are defined and for them an exchange formula holds (|3|).

In this paper we are going to define the space LG by using the suitable family of seminorms. This will imply some properties of the space LG′.

We shall also give the expansions of some of the important generalized functions from LG′ by using the convolution and Laplace transform in the space LG′.

1. We introduce in this paper the space LG by the following definition:

DEFINITION 1. *The space* LG *is the subspace of* $L^2(0,\infty) \cap C^\infty(0,\infty)$ *for which elements*

$$r_k(\phi) := \Big(\int_0^\infty |R^k \phi(x)|^2 dx \Big)^{1/2} < \infty , \quad k \in N_o, \quad (R := R_o)$$

$$(R^k \phi, \ell_n) = (\phi, R^k \ell_n), \quad k \in N_o , \ n \in N_o,$$

hold.

(If $\phi, \psi \in L^2(0,\infty)$ *then* $(\phi,\psi) := \int_0^\infty \phi(t)\bar{\psi}(t)dt = <\phi, \bar{\psi}>$ *).*

The convergence structure in this space is given by the sequence of seminorms (r_k), $k \in N_o$.

THEOREM 1. (i) *A function* ϕ *from* $L^2(0,\infty)$ *of the form*

$$\phi \overset{2}{=} \sum_{n=o}^{\infty} a_n \ell_n \qquad (\overset{2}{=} \text{ means: in square mean}) \tag{2}$$

is in LG *iff for every* $k \in N_o$ *there exists* $C_k > 0$ *such that*

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(3) $$n^k|a_n| \leq C_k \ , \quad n \in N_o$$

(ii) *A sequence* (ϕ_m) *from* $L^2(0,\infty)$ *of the form*

(4) $$\phi_m \stackrel{2}{=} \sum_{n=o}^{\infty} a_{m,n}\ell_n$$

converges in LG *to* $\phi \in$ LG *of the form (2), iff for any* $k \in N$ *there exists* $C_k > 0$ *such that*

(5) $$n^k|a_{m,n}| \leq C_k \ , \ n \in N_o \quad \text{and}$$

(6) *for every* $n \in N_o$, $a_{m,n} \to a_n$ *as* $m \to \infty$.

P r o o f. (i) Since (3) holds iff for every $k \in N_o$

$$\sum_{n=o}^{\infty} n^{2k}|a_n|^2 < \infty \ ,$$

the assertion (i) directly follows from |4| Lemma 9.3.3. .

(ii) The conditions (5) and (6) are equivalent to the condition:

for every $k \in N_o$, $\sum_{n=o}^{\infty} n^{2k}|a_{m,n}-a_n|^2 \to 0$ as $m \to \infty$.

This condition is equivalent to $r_k(\phi_n-\phi) \to 0$ for every $k \in N$.

Let us put $\tilde{D}^{2k}\phi := D(\underbrace{xD(xD(\ldots(xD}_{k-1}\phi)\ldots)))$; for example $\tilde{D}^2\phi = D(xD\phi)$, $\tilde{D}^4\phi = D(xD(xD\phi))$ $(\tilde{D}^o\phi = \phi)$.

We are going to prove the following theorem:

THEOREM 2. (i) *A function* $\phi \in$ LG *iff* $\phi \in C^\infty[0,\infty)$ *and*

$$\gamma_k(\phi) := \sup\{x^k[D^\ell\phi(x)]; \ x \in 0,\infty), \ \ell \leq k\} < \infty, \ k \in N_o \ ;$$

$$\rho_{2k}(\phi) := \sup\{[\tilde{D}^{2k}\phi(x)]; \ x \in 0,\infty)\} < \infty. \ , \quad k \in N \ .$$

(ii) *A sequence* (ϕ_m) *from* LG *converges to* 0 *iff for every* $k \in N_o$ $\gamma_k(\phi_m) \to 0$ *and* $\rho_{2k}(\phi_m) \to 0$ *as* $m \to \infty$.

First we are going to prove two lemmas.

LEMMA 3. *For the functions* (ℓ_n), $n \in N_o$, *the following formulas and estimates hold:*

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(i) *If* $k,j \in N_0$ *and* $j \le k$ *then*

$$x^k D^j \ell_n(x) = \sum_{i=0}^{2k} c_{n,k,j,i}\, \ell_{n-k+i}, \quad n \in N_0$$

and there exists $C_{k,j} > 0$ *such that*

$$|c_{n,k,j,i}| \le C_{k,j} \hat{n}^k, \quad i = 0,\dots,k \quad \left(\hat{n} = \begin{cases} n & n \neq 0 \\ 1 & n = 0 \end{cases}\right).$$

(ii) *If* $k \in N$

$$\tilde{D}^{2k} \ell_n = \sum_{i=0}^{2k} \tilde{c}_{n,k,i}\, \ell_{n-k+i}, \quad n \in N_0$$

and there exists $\tilde{C}_k > 0$ *such that*

$$|\tilde{c}_{n,k,i}| \le \tilde{C}_k \hat{n}.$$

P r o o f. From the formula from |1|, p. 188. for $\alpha = 0$ and $n \in N_0$, we have

$$(7)\quad x\ell_n(x) = -(n+1)\ell_{n+1}(x) + (2n+1)\ell_n(x) - n\ell_{n-1}; \quad (\ell_{-1} = 0)$$

$$(8)\quad xD\ell_n(x) = \frac{1}{2}(n+1)\ell_{n+1} - \frac{1}{2}\ell_n - \frac{n}{2}\ell_{n-1}.$$

So, by induction, one can prove (i).

For the assertion of (ii) we ought to combine (7), (8) with (1) for $\alpha = 0$. In this way we obtain

$$(9)\quad D(xD\phi) = -\frac{1}{4}(n+1)\ell_{n+1} - \frac{1}{4}(2n+1)\ell_n - \frac{1}{4}n\ell_{n-1}, \quad n \in N_0,$$

from which (ii) follows by induction.

LEMMA 4. *If* $\phi \in LG$ *is of the form*

$$\phi \overset{2}{=} \sum_{n=0}^{\infty} a_n \ell_n$$

then this series converges uniformly to ϕ *in* $[0,\infty)$.

Moreover, for any $k \in N_0$ *and* $j \in N_0$

$$(10)\quad x^k D^j \phi(x) = \sum_{n=0}^{\infty} a_n x^k D^j \ell_n(x)$$

In (10) *the series converges uniformly in* $[0,\infty)$.

P r o o f. For the proof we need the following formula and estimate from |1|.

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(11) $$DL_n^{\alpha} = - L_{n-1}^{\alpha+1} , \quad n \in N_o$$

(12) $$|L_n^{\alpha}(x)| \leq \frac{\Gamma(\alpha+n)}{\Gamma(\alpha)} \cdot \frac{1}{n!} e^{x/2}, \quad x \in [0,\infty), \ \alpha \geq 0, \ n \in N_o .$$

From (12) for $\alpha=0$ and $x \in [0,\infty)$, we have

$$\sum_{n=o}^{\infty} |a_n||\ell_n(x)| \leq \sum_{n=o}^{\infty} |a_n| < \infty .$$

This means that $\sum_{n=o}^{\infty} a_n \ell_n$ converges uniformly in $[0,\infty)$ to ϕ.

Let us prove (10) for $(k,j) = (1,0)$ and $(k,j) = (0,1)$ because for arbitrary (k,j), (10) follows by induction.

From (11) and (12) it follows that for every $n \in N_o$

$$|D\ell_n| = |e^{-x/2} (- \frac{1}{2} L_n(x) - L_{n-1}^1(x))| \leq n+1, x \in [0,\infty),$$

holds.

Since for (a_n), $n \in N_o$, (3) holds, we obtain that $\sum_{n=o}^{\infty} a_n \ell_n'$ converges uniformly on $[0,\infty)$.

Similarly, from (7) and (3) it follows that $\sum_{n=o}^{\infty} a_n x\ell_n(x)$ converges uniformly in $[0,\infty)$.

P r o o f of Theorem 2. (i) Let $\phi \in C^{\infty}[0,\infty)$ and for every $k \in N_o$, $\gamma_k(\phi) < \infty$, and $\rho_{2k}(\phi) < \infty$. Since the functions $x \mapsto x^k D^j \phi(x)$, $x \mapsto \tilde{D}^{2k}\phi(x)$, $j,k \in N_o$, $j \leq k$, are bounded on $[0,1)$, we have

(13) $$\int_o^{\infty} |x^k D^j \phi(x)|^2 dx \leq \int_o^1 |x^k D^j \phi(x)|^2 dx + \int_1^{\infty} \frac{1}{x^2} |x^{k+1} D^j \phi(x)|^2 dx < \infty ,$$

(14) $$\int_o^{\infty} |\tilde{D}^{2k}\phi(x)|^2 dx \leq \int_o^1 |\tilde{D}^{2k}\phi(x)|^2 dx + \int_1^{\infty} \frac{1}{x^2} |\tilde{D}^{2k}\phi(x)|^2 dx < \infty .$$

So we obtain that $x^k D^j \phi(x)$, $\tilde{D}^{2k}\phi(x) \in L^2(0,\infty)$, $j,k \in N_o$, $j \leq k$.

We may prove by induction that $R^k\phi$, $k \in N_o$, is a linear combination of factors of the form $x^r D^i \phi(x)$, $r,i \in N_o$, $i \leq r \leq k$, and $\tilde{D}^{2s}\phi(x)$, $s \in N$, $s \leq k$. So $R^k\phi \in L^2(0,\infty)$ for every $k \in N_o$. By induction we may, as well, prove that for every $k \in N_o$ and $n \in N_o$

$$(R^k\phi, \ell_n) = \int_o^{\infty} R^k\phi(t)\ell_n(t)dt = \int_o^{\infty} \phi(t) R^k \ell_n(t) dt , \quad \phi \in LG .$$

From Definition 1 it follows that $\phi \in LG$. If $\phi \in L^2(0,\infty)$ such that ϕ is of the form (2) and (3) holds, then from Lemma 4, using (7), (11) and (12), we obtain $\gamma_k(\phi) < \infty$ and $\rho_{2k}(\phi) < \infty$, $k \in N_o$.

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(ii) From (13) and (14) with ϕ_m instead of ϕ we have

$$x^k D^j \phi_m(x) \overset{2}{\to} 0, \quad \tilde{D}^{2k}\phi_m(x) \overset{2}{\to} 0, \quad i,k \in N_o, \ j \leq k, \text{ as } m \to \infty .$$

This means that for any $k \in N_o$, $R^k \phi_m \overset{2}{\to} 0$ as $m \to \infty$. If ϕ_m is of the form (4), $m \in N$, is a sequence from LG which converges to $0 \in LG$, using (5), (6), Lemma 4, (7), (11) and (12) we obtain that

$$\gamma_k(\phi_m) \to 0 \quad \text{and} \quad \rho_{2k}(\phi_m) \to 0, \ k \in N_o, \text{ as } m \to \infty .$$

We see that we do not need Lemma 3 for the proof of Theorem 2. Yet this Lemma and the appropriate assertion for the transposed mappings directly imply:

THEOREM 5. *The mappings from LG′ to LG′ defined by*

(15)
$$f \to D^j(x^k f) \quad j,k \in N_o, \ j \leq k;$$
$$f \to D(x(D^2(x(\ldots D^2(x(D^2(x(Df))))\ldots)))))$$

are continuous with respect to the strong (weak) topologies in LG′.

Let as remark that the convergence structure of LG′ is investigated in |2|.

2. Theorem 2 implies that LG is a subspace of $E(0,\infty)$ and that the convergence in LG implies the convergence in $E(0,\infty)$. Since $\mathcal{D}(0,\infty) \subset LG$ and $\mathcal{D}(0,\infty)$ is a dense subspace of $E(0,\infty)$ we have

$$E'(0,\infty) \subset LG' \quad \text{(see also |4| p. 319.)}.$$

If $F \in L^1_{loc}(0,\infty)$ and for some $k \in N$, $F(t)t^{-k} \in L^1(1,\infty)$ then by

$$\phi \to \langle F,\phi\rangle := \int_0^\infty F(t)\phi(t)dt = \int_0^1 F(t)\phi(t)dt + \int_1^\infty F(t)t^{-k}(t^k\phi(t))dt$$

an element from LG′ is defined.

So Theorem 5 and Theorem 9.6.2. from |4| implies

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THEOREM 6. (i) |4| $f \in LG'$ *iff* f *is of the form*

$$f = R^k F + c_o \exp(-x/2)$$

for some $F \in L^2(0,\infty)$, $k \in N$ *and* c_o*-complex number.*

(ii) *If* $F \in L^1_{loc}(0,\infty)$ *and* $F(t)t^{-k} \in L^1(1,\infty)$ *for some* $k \in N$ *then*

$$\tilde{D}^{2k}F \quad \text{and} \quad D^j x^r F$$

are from LG' *for every* $k \in N$ *and every* $j,r \in N_o$, $j \leq r$.

There is an open problem concerning the connections between the space LG' and $S'(0,\infty)$.

3. In |3| the convolution of elements $f = \sum_{n=o}^{\infty} a_n \ell_n$ and $g = \sum_{n=o}^{\infty} b_n \ell_n$ from LG' is defined by

$$f \Theta g = \sum_{n=o}^{\infty} c_n \ell_n$$

where $c_n = \sum_{p+q=n} a_p b_q - \sum_{p+q=n-1} a_p b_q$ ($\sum_{p+q=-1} = 0$).

With this operation, LG' is a convolution algebra and in |3| we have proved that the mapping from $LG' \times LG'$ into LG' defined by

$$(f,g) \to f \Theta g$$

is sequentially continuous (in the sense of weak topology).

If $f \in L^2$ and $g \in L^2$ then in LG'

$$(f \Theta g)(x) = \int_o^{\infty} f(x-t)g(t)dt$$

holds (|3|).

The Laplace transform in LG' is defined by

$$\mathcal{L}(f) := \langle f, e^{-st} \rangle = \langle \sum_{n=o}^{\infty} a_n \ell_n, e^{-st} \rangle = \sum_{n=o}^{\infty} a_n \mathcal{L}(\ell_n)$$

$$= \sum_{n=o}^{\infty} a_n \frac{(s-1/2)^n}{(s+1/2)^{n+1}}, \quad \text{Res} > 0 \text{ (see |4|).}$$

We have shown that $\mathcal{L}$ is a sequentially continuous bijection of LG' (in relation to the weak topology in LG') and the space of functions L, analytic in the half-space Res > 0 of the form

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$$F(s) = \sum_{n=0}^{\infty} a_n \frac{(s-1/2)^n}{(s+1/2)^{n+1}}$$

such that coefficients a_n, $n \in N_0$, satisfy the condition

$$|a_n| \leq M n^k$$

for some $M > 0$ and $k \in N$. In $\mathcal{L}$ we have a topology induced by the topology of the uniform convergence on a compact subsets of the half-space $\mathrm{Re}\, s > 0$.

We have shown, as well in [3], that

$$\mathcal{L}(f \Theta g) = \mathcal{L}(f)\mathcal{L}(g) .$$

Let $f = \sum_{n=0}^{\infty} c_n \ell_n \in L^2(0,\infty)$ and $f_\nu = \sum_{n=0}^{\nu} c_n \ell_n$. Since $e^{-st} \in L^2(0,\infty)$, for any $s = \xi + i\eta$, $\xi > 0$, in the sense of the uniform convergence on compact subsets on the half-space $\mathrm{Re}\, s > 0$, we have

$$\sum_{n=0}^{\nu} a_n \frac{(s-1/2)^n}{(s+1/2)^{n+1}} = \int_0^{\infty} e^{-st} f_\nu(t)dt \to \int_0^{\infty} e^{-st} f(t)dt = \mathcal{L}(f)(s)$$

as $\nu \to \infty$.

This means that the Laplace transformation in LG' is a generalization of the Laplace transformation in $L^2(0,\infty)$.

4. We are going to give the expansions into a series of some elements from LG' in the sense of the convergence in LG'.

Since

$$\int_0^{\infty} e^{-st} t^{\alpha} dt = \frac{\Gamma(\alpha+1)}{s^{\alpha+1}} , \quad \mathrm{Re}\, s > 0, \ \alpha \neq -1 , \tag{16}$$

we have (for $s = \frac{1}{2}$) that $\int_0^{\infty} \ell_n(t)dt = 2(-1)^n$. This implies the expansion of function $f(x) = 1$, $x \in [0,\infty)$:

$$1 = 2 \sum_{n=0}^{\infty} (-1)^n \ell_n , \tag{17}$$

Let the δ-distribution $\delta(t-x)$, $x \geq 0$, be defined by $\langle \delta(t-x), \phi(t) \rangle := \phi(x)$. It is clear that $\delta(t-x) \in LG'$. We are going to find the coefficients a_n, $n \in N_0$, in the expansion

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$$\delta(t-x) = \sum_{n=0}^{\infty} a_n \ell_n(t).$$

We have

(18) $$\mathcal{L}(\delta(t-x))(s) = \langle \delta(t-x), e^{-st} \rangle = e^{-xs}$$

(19) $$\mathcal{L}(\delta(t-x))(s) = \sum_{n=0}^{\infty} a_n \mathcal{L}(\ell_n(t)) = \sum_{n=0}^{\infty} a_n \frac{(s-1/2)^n}{(s+1/2)^{n+1}}$$

In LG' we have

$$e^{-xs} = \sum_{n=0}^{\infty} \frac{(s-1/2)^n}{(s+1/2)^{n+1}} \ell_n(x)$$

Thus from (18) and (19) we obtain

$$\sum_{n=0}^{\infty} a_n \frac{(s-1/2)^n}{(s+1/2)^{n+1}} = \sum_{n=0}^{\infty} \frac{(s-1/2)^n}{(s+1/2)^{n+1}} \ell_n(x).$$

Since $\mathcal{L}$ transformation is a bijection from LG' to L, we have

(20) $$\delta(t-x) = \sum_{n=0}^{\infty} \ell_n(x) \ell_n(t).$$

Particularly, because $\ell_n(0) = 1$, $n \in N_o$, for $x = 0$, we have $\delta(t) = \sum_{n=0}^{\infty} \ell_n(t)$. If $f(t) \in LG'$ and $x \geq 0$, let us define the operation of the translation in LG' by:

$$\langle f(t-x), \phi(t) \rangle := \langle f(t), \phi(t+x) \rangle \quad , \quad \phi \in LG .$$

Since the mapping form LG to LG defined by

$$\phi(t) \to \phi(x+t)$$

is continuous, it follows that $f(t-x) \in LG'$. If $f \in LG'$ is a function in $(0,\infty)$, then a generalized function $f(t-x)$ is a function which is equal to zero in $[0,x]$ and to $f(t-x)$ in (x,∞).

Let us prove: $\delta(t-x) \Theta f(t) = f(t-x)$. In order to prove this formula, some formulas for the Laguerre polynomials are needed.

LEMMA 7. *For $x \geq 0$ and $t \in [0,\infty)$ we have*

(21) $$L_n(t+x) = \sum_{p+q=n} L_p(x) L_p(t) - \sum_{p+q=n-1} L_p(x) L_q(t),$$

$$\left(\sum_{p+q=-1} = 0 \right).$$

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P r o o f. The coefficient of t^{ℓ}, $0 \le \ell \le n$, on the left side of (21) is

$$(22) \qquad \sum_{r=0}^{n-\ell} \binom{n}{\ell+r} \binom{\ell+r}{r} \frac{(-1)^{\ell+r}}{(\ell+r)!} x^r .$$

The coefficient of t^{α} on the right side of (21) is

$$(23) \qquad \frac{(-1)^{\ell}}{\ell!} \left[\sum_{r=0}^{n-\ell-1} \sum_{i=0}^{n-(\ell+r)} \binom{n-(\ell+r)}{i} \frac{(-1)^i x^i}{i!} \binom{\ell+r}{\ell} - \right.$$

$$\left. - \sum_{r=0}^{n-1-(\ell+r)} \binom{n-1-(\ell+r)}{i} \frac{(-1)^i x^i}{i!} \binom{\ell+r}{\ell} \right] .$$

The coefficients of x^i in (22) and (23) are:

$$(24) \qquad \binom{n}{\ell+i} \frac{(-1)^{\ell+i}}{(\ell+i)!} \binom{\ell+i}{i}$$

$$(25) \qquad \frac{(-1)^{\ell}}{\ell!\, i!} \sum_{r=0}^{n-\ell-1} \binom{n-\ell-r}{i} \binom{\ell-1+r}{\ell-1} .$$

The equality of (24) and (25) follows from the known identity:

$$\sum_{r=0}^{n-\ell-i} \binom{\ell-1+r}{\ell-1} \binom{n-\ell-r}{i} = \binom{n}{\ell+i} .$$

From this Lemma, by a simple calculation of the coefficients of $f(t-x)$ and using (20), we obtain

$$(26) \qquad \delta(t-x) \odot f(t) = f(t-x)$$

From (17) and (26) we have:

$$h(t-x) := \begin{cases} 0 & 0 \le t \le x \\ 1 & t > x \end{cases} = \sum_{n=0}^{\infty} \left(\sum_{i=0}^{n-1} 4(-1)^{n-i} \ell_i(x) + 2\ell_n(x) \right) \ell_n(t).$$

From (26) and the identity

$$\int_0^{\infty} e^{-st} x^m L_n(t)\,dt = (-1)^{m+n} \frac{d^m}{ds^m} \left[(1-1/s)\, 1/s \right] , \quad n \in N_0, \; m \in N_0,$$

one may easily obtain the expansions of the generalized functions

$$(t-x)_+^m = \begin{cases} 0 & 0 \le t \le x \\ (t-x)^m & t > x . \end{cases}$$

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REZIME

ANALITIČKA KARAKTERIZACIJA PROSTORA UOPŠTENIH FUNKCIJA KOJE IMAJU LAGEROVU EKSPANZIJU

Prostor uopštenih funkcija čiji elementi imaju Lagerovu ekspanziju u redove su ispitivani u |2| i |3|. U ovom radu smo definisali prostor L pomoću odgovarajuće familije seminormi. Na taj način smo dobili odgovarajuće osobine prostora LG' kao i teoremu o reprezentaciji elemenata iz LG'. Takodje, koristeći konvoluciju u LG' kao i Laplasovu transformaciju dajemo razlaganje u redove važnih uopštenih funkcija iz LG'.

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ON HERMITE POLYNOMIALS OF THE GAUSSIAN RANDOM PROCESS

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ABSTRACT

An explicit expression is defined to be a Hermite polynomial of degree n, n=1,2,..., in variables $\xi(t_1)$, $\xi(t_2)$,...,$\xi(t_n) \in \{\xi(t), t \in T\}$, where $\{\xi(t), t \in T\}$ is a real Gaussian process. Some properties of these polynomials are investigated. Especially, $E^s H_n(\xi_1,\xi_2,\ldots,\xi_n) = H_n(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n)$, where $\hat{\xi}_k = E^s \xi_k$.

Let $\{\xi(t), t \in T\}$ be a real Gaussian random process on a probability space (Ω,F,P), with the expectation $E\xi(t) = 0$, $t \in T$ and covariance function $E\xi(t_i)\xi(t_j) = B(t_i,t_j)$, $t_i,t_j \in T$. Let $\boldsymbol{H}(T)$ be Hilbert space of square integrable functions, measurable with respect to the σ-field $F(T)$, generated by $\{\xi(t), t \in T\}$, with the scalar product

$$(\phi,\psi) = E\phi\psi \ , \quad \phi,\psi \in \boldsymbol{H}(T) \ .$$

We take that $E\phi = 0$, $\phi \in \boldsymbol{H}(T)$.

In |4| a Hermite polynomial of degree n, in variables $\xi(t_1),\xi(t_2),\ldots,\xi(t_k)$, $t_1,\ldots,t_k \in T$ is defined as a polynomial orthogonal to all polynomials of a degree less than n. It is shown that

$$\boldsymbol{H}(T) = \sum_{p=1}^{\infty} \oplus H_p(T) \ ,$$

where $H_p(T)$ is a linear closure spanned by Hermite polynomials of degree p, in variables $\xi(t)$, $t \in T$. In |3|, explicit expressions for Hermite polynomials of degree 1, 2, and 3 are given.

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In this paper an explicit expression is defined as a Hermite polynomial of degree n, ,n=1,2,... and some properties of these polynomials are investigated (Theorem 3, 4, 5).

A well known formula for the moments of the Gausiian n-dimensional random variable $(\xi(t_1),\xi(t_2),\ldots,\xi(t_n))$, where n is even, will be used

$$(1)\qquad E\xi(t_1)\xi(t_2)\ldots\xi(t_n) = \Sigma\prod B(t_i,t_j)\ ,$$

where the sum ranges over all the possible partitions of the set $\{t_1,t_2,\ldots,t_n\}$ to pairs $\{t_i,t_j\}$ and the product ranges over all the pairs $\{t_i,t_j\}$ of a corresponding partition.

DEFINITION *A Hermite polynomial, of degree* n *of* n *variables, is*

$$(2)\quad H_n(\xi_1,\xi_2,\ldots,\xi_n) = \xi_1\xi_2\ldots\xi_n - \Sigma B_{i_1j_1}\alpha_{i_1j_1}(\xi_1,\ldots,\xi_n) +$$

$$+ \Sigma B_{i_1j_1}B_{i_2j_2}\alpha_{i_1j_1i_2j_2}(\xi_1,\ldots,\xi_n)+\ldots+(-1)^k\Sigma B_{i_1j_1}\ldots$$

$$\ldots B_{i_kj_k}\alpha_{i_1j_1\ldots i_kj_k}(\xi_1,\ldots,\xi_n) + \ldots + (-1)^{\left[\frac{n}{2}\right]}\Sigma\prod B_{ij}\ ,$$

$$\xi_i = \xi(t_i),\ B_{i_\nu j_\nu} = B(t_{i_\nu},t_{j_\nu}),\ i_\nu,j_\nu \in \{1,2,\ldots,n\}\ .$$

We take that $i_\nu < j_\nu$ which is not a restriction,

$$\alpha_{i_1j_1\ldots i_kj_k}(\xi_1,\ldots,\xi_n) = \prod_{i=1}^{n}\xi_i,\quad i\neq i_\nu,\quad i\neq j_\nu,\ \nu=1,2,\ldots,k,$$

$$k=1,2,\ldots,\left[\frac{n}{2}\right].$$

The first sum, in (2), ranges over all the possible choices of a pair (i_1,j_1) from the set $1,2,\ldots,n$. The k-th sum, $k=2,3,\ldots,\left[\frac{n}{2}\right]-1$, in (2), ranges over all the possible choices of k pairs $(i_1,j_1),(i_2,j_2),\ldots,(i_k,j_k)$, from the set $\{1,2,\ldots,n\}$, where the order of pairs is not important. The last term in (2) is $\Sigma\prod B_{ij}$, where the sum ranges over all the possible partitions of the set $\{1,2,\ldots,n\}$ to pairs (i,j) and the product ranges over all the pairs of a corresponding partition. Notice that $\Sigma\prod B_{ij}$ is equal to $E\xi_1\xi_2\ldots\xi_n$, given in (1).

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$\Sigma B_{i_1 j_1} \alpha_{i_1 j_1}(\xi_1,\ldots,\xi_n$ has $\binom{n}{2}$ terms,

$\Sigma B_{i_1 j_1} B_{i_2 j_2} \alpha_{i_1 j_1 i_2 j_2}(\xi_1,\ldots,\xi_n)$ has $\frac{1}{2}\binom{n}{2}\binom{n-2}{2} = \binom{n}{4} 3!!$ terms,

$\Sigma B_{i_1 j_1} \ldots B_{i_k j_k} \alpha_{i_1 j_1 \ldots i_k j_k}(\xi_1,\ldots,\xi_n)$ has

$\frac{1}{k!}\binom{n}{2}\binom{n-2}{2}\ldots\binom{n-2(k-1)}{2} = \binom{n}{2k}(2k-1)!!$ terms because there is

$\binom{n}{2}\binom{n-2}{2}\ldots\binom{n-2(k-1)}{2} = A$ ways to choose k pairs from the set $\{1,2,\ldots,n\}$, and as the order of pairs is not important, the exact number is $\frac{1}{k!} A$

$\Sigma \prod B_{ij}$ has $(n-1)!!$ terms.

The polynomial $H_n(\xi_1,\ldots,\xi_n)$, for an n even, contains only elements of an even degree and the constant element.

For n odd, it contains only elements of an odd degree and the constant element is zero. In $H_n(\xi_1,\ldots,\xi_n)$ we may have some equal variables. So we have a Hermite polynomial in one, two, there, e.t.c. variables of degree n. If $\xi_1 = \xi_2 = \ldots = \xi_n$, we have the Hermite polynomial of one variable. For example, $H_1(\xi) = \xi$, $H_2(\xi) = \xi^2 - \sigma^2$, $H_3(\xi) = \xi^3 - 3\sigma^2\xi$, $H_4(\xi) = \xi^4 - 6\sigma^2\xi^2 + 3\sigma^4$, $H_5(\xi) = \xi^5 - 10\sigma^2\xi^3 + 15\sigma^4\xi$. In general

$$H_n(\xi) = \xi^n + \sum_{k=1}^{[\frac{n}{2}]} (-1)^k (2k-1)!! \binom{n}{2k} \sigma^{2k} \xi^{n-2k},$$

where $\xi : N(0,\sigma^2)$.

If $\xi(t)$, $t \in T$ are independent random variables, then $H_n(\xi_1,\ldots,\xi_k) = H_{n_1}(\xi_1) H_{n_2}(\xi_2) \ldots H_{n_k}(\xi_k)$, for $k \leq n$, $n = n_1 + n_2 + \ldots + n_k$. $H_{n_i}(\xi_i)$, $i=1,2,\ldots,k$, are the Hermite polynomials of one variable, which is a well known definition of the Hermite polynomial of degree n, in k variables, see for instance [1].

THEOREM 1.

$$E\, H_n(\xi_1,\ldots,\xi_n) = 0.$$

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P r o o f. The statement follows immediately for n odd because all odd moments of the Gaussian n-dimensional random variable $(\xi_1,\dots,\xi_n)$ are zero. Let $n = 2m$, $m = 1,2,\dots$. Then

$$(3)\quad EH_n(\xi_1,\dots,\xi_n) = E\xi_1\xi_2\dots\xi_n - \Sigma B_{i_1j_1} E\alpha_{i_1j_1}(\xi_1,\dots,\xi_n) + \dots +$$
$$+ (-1)^k \Sigma B_{i_1j_1}\dots B_{i_kj_k} E\alpha_{i_1j_1}\dots i_kj_k(\xi_1,\dots,\xi_n) + \dots +$$
$$+ (-1)^m \Sigma \prod B_{ij} .$$

Let us notice an arbitrary term $B_{i_{1o}j_{1o}} B_{i_{2o}j_{2o}} \dots B_{i_{mo}j_{mo}}$ in the first member $E\xi_1 \dots \xi_n$ of (3). This term occurs in the second member of (3) exactly $m = n/2$ times, because $B_{i_1j_1}$ can be any one of $B_{i_{1o}j_{1o}},\dots,B_{i_{2o}j_{2o}},\dots,B_{i_{mo}j_{mo}}$. It occurs $\binom{m}{k}$ times in the k-th member of (3), $k = 2,\dots,m-1$, because $B_{i_1j_1},B_{i_2j_2},\dots,B_{i_kj_k}$ can be any k elements from $B_{i_{1o}j_{1o}},B_{i_{2o}j_{2o}},\dots,B_{i_{mo}j_{mo}}$. In the last member $\Sigma \prod B_{ij}$ it occurs only once. According this, the coefficient of $B_{i_{1o}j_{1o}} B_{i_{2o}j_{2o}} \dots B_{i_{mo}j_{mo}}$ in (3) is

$$1 - m + \binom{m}{2} + \dots + (-1)^k \binom{m}{k} + \dots + (-1)^m \binom{m}{m} = 0.$$

Since the product $B_{i_{1o}j_{1o}} \dots B_{i_{mo}j_{mo}}$ was arbitrary, the same is valid for any other term in $E\xi_1 \dots \xi_n$.

THEOREM 2. *The Hermite polynomial* $H_n(\xi_1,\dots,\xi_n)$ *is orthogonal to every polynomial in* $H(T)$ *of degree* k, $k < n$.

P r o o f. It is sufficient to see that

$$E\, H_n(\xi_1,\dots,\xi_n)\eta_1\eta_2 \dots \eta_k = 0,$$

where $\eta_i = \eta(t_i) \in \{\xi(t),\ t \in T\}$, $i = 1,2,\dots,k$, $k < n$. If n is even and k odd, or conversely, the statement immediately follows, because, after multiplication, we get a polynomial with all the elements of odd degree, so, its expectation is zero. Let n and k be both odd, or both even. We shall show the case $n = 2m$, $k = 2q$, $m,q \in N$.

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The proof is analogous when n and k are both odd. Denote

$$B(i_s,i_p) = E\xi_{i_s}\xi_{i_p}, \quad i_s,i_p \in\{1,2,\dots,n\},$$

$$B(i_s,j_p) = E\xi_{i_s}\eta_{j_p}, \quad i_s \in\{1,2,\dots,n\},\ j_p \in \{1,2,\dots,k\},$$

$$B(j_s,j_p) = E\eta_{j_s}\eta_{j_s}, \quad j_s,j_p \in\{1,2,\dots,k\}.$$

The terms in $E\xi_1 \dots \xi_n\eta_1 \dots \eta_k$ have the form

$$(4)\quad B(i_1,i_2)\dots B(i_{\nu-1},i_\nu)B(i_{\nu+1},j_1)\dots B(i_n,j_\mu)B(j_{\mu+1},j_{\mu+2})\dots$$

$$\dots B(j_{k-1},j_k),\ \nu\in\{2,4,\dots,n\},\ \mu=n-\nu,\ \mu\le k.$$

For $\nu=n$, $(\mu=0)$ we have

$$B(i_1,i_2)\dots B(i_{n-1},i_n)B(j_1,j_2)\dots B(j_{k-1},j_k).$$

For $\nu=n-k$, $(\mu=k)$ we have

$$B(i_1,i_2)\dots B(i_{n-k-1},i_{n-k})B(i_{n-k+1},j_1)\dots B(i_n,j_k).$$

Let us notice an arbitrary term in $E\xi_1\dots\xi_n\eta_1\dots\eta_k$ as it is given in (4). In $\Sigma B_{i_1j_1} E\alpha_{i_1j_1}(\xi_1,\dots,\xi_n)\eta_1\dots\eta_k$ the term (4) occurs $\binom{\nu/2}{1} = \nu/2$ times, because $B_{i_1j_1}$ can be any one of the $B(i_1,i_2),\dots,B(i_{\nu-1},i_\nu)$. In $\Sigma B_{i_1j_1}\dots B_{i_rj_r} E\alpha_{i_1j_1\dots i_rj_r}(\xi_1,\dots,\xi_n)\eta_1\dots\eta_k$, for $r\le\nu/2$, the term (4) occurs $\binom{\nu/2}{r}$ times, and for $r>\nu/2$ it does not occur. We have that in $EH_n(\xi_1,\dots,\xi_n)\eta_1\dots\eta_k$ the term (4) occurs

$$1-\binom{\nu/2}{1}+\dots+(-1)^r\binom{\nu/2}{r}+\dots+(-1)^{\nu/2}\binom{\nu/2}{\nu/2} = 0$$

times. Since the term (4) is arbitrary, this is valid for any other term in $E\xi_1\dots\xi_n\eta_1\dots\eta_k$.

REMARK 1. Let $S_1,S_2\subseteq T$ be arbitrary and $H_p(S_1),H_q(S_2)$ the corresponding linear closures, spanned by the Hermite polynomial of degree p, in the variables $\xi(t_1),\dots,\xi(t_k)$, $t_1,\dots,t_k \in S_1$ and of degree q in the variables $\xi(t_1),\dots,\xi(t_j)$, $t_1,\dots,t_j \in S_2$. In [4] it is shown that

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$$H_p(S_1) \perp H_q(S_2) , \quad p \neq q.$$

The proof follows also from Theorem 2., because $\xi_1,\ldots,\xi_n,\eta_1,\ldots,\eta_k$ are the arbitrary elements from $\{\xi(t),\ t \in T\}$.

THEOREM 3. *The Hermite polynomial* $H_n(\xi_1,\ldots,\xi_n)$, n=2, 3,... *satisfies the partial differential equations*

$$\frac{\partial}{\partial \xi_k} H_n(\xi_1,\ldots,\xi_n) = H_{n-1}(\xi_1,\ldots,\xi_{k-1},\xi_{k+1},\ldots,\xi_n),$$

$$k = 1,2,\ldots,n .$$

P r o o f. Denote

$$\sum_{i_1,j_1,\ldots,i_r,j_r \in \{1,\ldots,n\}} B_{i_1j_1}\ldots B_{i_rj_r}\alpha_{i_1j_1\ldots i_rj_r}(\xi_1,\ldots,\xi_n)$$

by

$$\sum_{1,2,\ldots,n} B_{i_1j_1}\ldots B_{i_rj_r}\alpha_{i_1j_1\ldots i_rj_r}(\xi_1,\ldots,\xi_n) .$$

$$\frac{\partial}{\partial \xi_1} H_n(\xi_1,\ldots,\xi_n) = \xi_2\ldots\xi_n - \frac{\partial}{\partial \xi_1}\sum B_{i_1j_1}\alpha_{i_1j_1}(\xi_1,\ldots,\xi_n) +$$

$$+\ldots+(-1)^r \frac{\partial}{\partial \xi_1}\sum B_{i_1j_1}\ldots B_{i_rj_r}\alpha_{i_1j_1\ldots i_rj_r}(\xi_1,\ldots,\xi_n) +$$

$$+ \ldots+ (-1)^{\left[\frac{n}{2}\right]} \frac{\partial}{\partial \xi_1} \sum \prod B_{ij} .$$

$$\frac{\partial}{\partial \xi_1} \sum_{1,\ldots,n} B_{i_1j_1}\alpha_{i_1j_1}(\xi_1,\ldots,\xi_n) = \sum_{2,\ldots,n} B_{i_1j_1}\alpha_{i_1j_1}(\xi_2,\ldots,\xi_n) .$$

The sum on the right hand side ranges over all the choices of a pair (i_1,j_1) from the set $\{2,\ldots,n\}$, because after differentiation all the terms with coefficient $B_{1j}, j \in \{2,\ldots,n\}$ disappear, namely, the corresponding products $\alpha_{1j}(\xi_1,\ldots,\xi_n)$ do not contain ξ_1. In the rest of the terms $B_{ij}\alpha_{ij}(\xi_1,\ldots,\xi_n)$, (i,j) are chosen from the set $\{2,\ldots,n\}$ and

$$\frac{\partial}{\partial \xi_1}\alpha_{ij}(\xi_1,\ldots,\xi_n) = \alpha_{ij}(\xi_2,\ldots,\xi_n) .$$

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$$\frac{\partial}{\partial \xi_1} \sum_{1,\ldots,n} B_{i_1 j_1} \ldots B_{i_r j_r} \alpha_{i_1 j_1, \ldots i_r j_r}(\xi_1, \ldots, \xi_n) =$$

$$\sum_{2,\ldots,n} B_{i_1 j_1} \ldots B_{i_r j_r} \alpha_{i_1 j_1 \ldots i_r j_r}(\xi_2, \ldots, \xi_n)$$

The sum on the right hand side renges over all the choices of r pairs $(i_1,j_1),\ldots,(i_r,j_r)$, from the set $\{2,\ldots,n\}$, because all the terms which do not contain ξ_1 in $\alpha_{i_1 j_1 \ldots i_r j_r}(\xi_1,\ldots,\xi_n)$ disappear. In the other terms, the indexes are chosen from the set $\{2,\ldots,n\}$, and

$$\frac{\partial}{\partial \xi_1} \alpha_{i_1 j_1 \ldots i_r j_r}(\xi_1,\ldots,\xi_n) = \alpha_{i_1 j_1 \ldots i_r j_r}(\xi_2,\ldots,\xi_n).$$

This is independent of r, so it is valid for every term in $H_n(\xi_1,\ldots,\xi_n)$. The same is true if we consider any of $\xi_2,\ldots,\xi_n$ instead of ξ_1.

THEOREM 4. *The polynomials* $P_n(\xi_1,\ldots,\xi_n)$, $n \geq 2, P_1(\xi_1) = \xi_1$, *which satisfy the partial differential equations*

$$\frac{\partial}{\partial \xi_k} P_n(\xi_1,\ldots,\xi_n) = P_{n-1}(\xi_1,\ldots,\xi_{k-1},\xi_{k+1},\ldots,\xi_n)$$

$$k=1,2,\ldots,n,\ \xi_i = \xi(t_i),\ t_i \in T,\ i=1,2,\ldots,n$$

with the initial condition $EP_n(\xi_1,\ldots,\xi_n) = 0$ *are Hermite polynomials* $H_n(\xi_1,\ldots,\xi_n)$, $n=1,2,\ldots$.

Before the proof let us consider the following examples

1. $P_1(\xi_1) = \xi_1 = H_1(\xi_1)$

2. $\frac{\partial}{\partial \xi_1} P_2(\xi_1,\xi_2) = P_1(\xi_2) = \xi_2$, so $P_2(\xi_1,\xi_2) = \xi_1\xi_2 + \phi(\xi_2)$.

$$\frac{\partial}{\partial \xi_2} P_2(\xi_1,\xi_2) = P_1(\xi_1) = \xi_1 = \xi_1 + \phi'(\xi_2), \text{ so } \phi'(\xi_2) = 0,$$

and $P_2(\xi_1,\xi_2) = \xi_1\xi_2 + c$. The constant c is evaluated from the condition $EP_2(\xi_1,\xi_2) = 0$, and it is equal to $-E\xi_1\xi_2 = -B_{12}$, so finally

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$$P_2(\xi_1,\xi_2) = \xi_1\xi_2 - B_{12} = H_2(\xi_1,\xi_2) .$$

3. $\frac{\partial}{\partial\xi_1} P_3(\xi_1,\xi_2,\xi_3) = \xi_2\xi_3 - B_{23}$, so , $P_3(\xi_1,\xi_2,\xi_3) = \xi_1\xi_2\xi_3 - B_{23}\xi_1 + \phi_1(\xi_2,\xi_3)$

$\frac{\partial}{\partial\xi_2} P_3(\xi_1,\xi_2,\xi_3) = \xi_1\xi_3 + \frac{\partial}{\partial\xi_2}\phi_1(\xi_2,\xi_3)$, so , $\phi_1(\xi_1,\xi_2) = -B_{13}\xi_2 + \phi_2(\xi_3)$, and $P_3(\xi_1,\xi_2,\xi_3) = \xi_1\xi_2\xi_3 - B_{23}\xi_1 - B_{13}\xi_2 + \phi_2(\xi_3)$.

$\frac{\partial}{\partial\xi_3} P_3(\xi_1,\xi_2,\xi_3) = \xi_1\xi_2 - B_{12} = \xi_1\xi_2 + \frac{\partial}{\partial\xi_3}\phi(\xi_3)$, so , $\phi_2(\xi_3) = -B_{12}\xi_3 + c$.

From $EP_3(\xi_1,\xi_2,\xi_3) = 0$, it follows that $c = 0$ and finally

$$P_3(\xi_1,\xi_2,\xi_3) = \xi_1\xi_2\xi_3 - B_{23}\xi_1 - B_{13}\xi_2 - B_{12}\xi_3 = H_3(\xi_1,\xi_2,\xi_3) .$$

P r o o f. We know that $P_1(\xi_1) = \xi_1 = H_1(\xi_1)$ and suppose that

$$P_{n-1}(\xi_1,\ldots,\xi_{n-1}) = H_{n-1}(\xi_1,\ldots,\xi_{n-1}) .$$

Then

$$\frac{\partial}{\partial\xi_1} P_n(\xi_1,\ldots,\xi_n) = \frac{\partial}{\partial\xi_1} P_n = P_{n-1}(\xi_2,\ldots,\xi_n) = H_{n-1}(\xi_2,\ldots,\xi_n) =$$

$= \xi_2\xi_3\ldots\xi_n - \sum_{2,\ldots,n} B_{ij}\alpha_{ij}(\xi_2,\ldots,\xi_n) + \ldots$. It follows that

$$P_n = \xi_1\xi_2\ldots\xi_n - \sum_{2,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\ldots,\xi_n)\xi_1 + \ldots + \phi_1(\xi_2,\ldots,\xi_n).$$

$$\frac{\partial}{\partial\xi_2} P_n = \xi_1\xi_3\ldots\xi_n - \sum_{3,\ldots,n} B_{ij}\alpha_{ij}(\xi_3,\ldots,\xi_n)\xi_1 + \ldots + \frac{\partial}{\partial\xi_2}\phi_1(\xi_2,$$

$\ldots,\xi_n) = P_{n-1}(\xi_1,\xi_3,\ldots,\xi_n) = H_{n-1}(\xi_1,\xi_3,\ldots,\xi_n) =$

$= \xi_1\xi_3\ldots\xi_n - \sum_{1,3,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\xi_3,\ldots,\xi_n) + \ldots$, whence

$$\frac{\partial}{\partial\xi_2}\phi_1(\xi_2,\ldots,\xi_n) = \sum_{j=3}^{n} B_{1j}\alpha_{1j}(\xi_1,\xi_3,\ldots,\xi_n) + \ldots \text{ and}$$

$$P_n = \xi_1\xi_2\ldots\xi_n - \sum_{2,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\ldots,\xi_n) - \sum_{j=3}^{n} B_{1j}\alpha_{1j}(\xi_1,\ldots,\xi_n) + \phi_2(\xi_3,\ldots,\xi_n),$$

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$$\frac{\partial}{\partial \xi_3} P_n = \xi_1\xi_2\xi_3\cdots\xi_n - \sum_{2,4,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\xi_2,\xi_4,\ldots,\xi_n) -$$

$$- \sum_{j=4}^{n} B_{1j}\alpha_{1j}(\xi_1,\xi_2,\xi_4,\ldots,\xi_n) + \ldots + \frac{\partial}{\partial \xi_3}\phi_2(\xi_3,\ldots,\xi_n) =$$

$$= \xi_1\xi_2\xi_4\cdots\xi_n - \sum_{1,2,4,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\xi_2,\xi_4,\ldots,\xi_n) + \ldots,$$

whence

$$\frac{\partial}{\partial \xi_n}\phi_2(\xi_3,\ldots,\xi_n) = -B_{12}\alpha_{12}(\xi_1,\xi_2,\xi_4,\ldots,\xi_n) + \ldots \text{ , so}$$

$$P_n(\xi_1,\ldots,\xi_n) = \xi_1\cdots\xi_n - \sum_{2,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\ldots,\xi_n) -$$

$$- \sum_{j=3}^{n} B_{1j}\alpha_{1j}(\xi_1,\ldots,\xi_n) - B_{12}\alpha_{12}(\xi_1,\ldots,\xi_n) + \ldots + \phi_3(\xi_4,\ldots,\xi_n) =$$

$$= \xi_1\cdots\xi_n - \sum_{1,2,\ldots,n} B_{ij}\alpha_{ij}(\xi_1,\ldots,\xi_n) + \ldots + \phi_3(\xi_4,\ldots,\xi_n).$$

In the same way we get the third and the other elements in $P_n(\xi_1,\ldots,\xi_n) = H_n(\xi_1,\ldots,\xi_n)$.

Let $S \subset T$. Denote $F^S\xi(t) = E(\xi(t)\,|\,F(S))$. $t \in T$. Where $F(S)$ is the σ-algebra generated by $\{\xi(t),\ t \in S\}$.

In |2| it is shown that the conditional expectation, with respect to $F(S)$, of a polynomial of degree n, in variables $\xi(t_1)$, $\ldots,\xi(t_k)$, $t_1,\ldots,t_k \in T$, is a polynomial of degree n, in $\hat{\xi}(t_1),\ldots,\hat{\xi}(t_k)$, $t_1,\ldots,t_k \in T$, where $\hat{\xi}(t) = E^S\xi(t)$, $t \in T$.

THEOREM 5. $E^S H_n(\xi_1,\ldots,\xi_n) = H_n(\hat{\xi}_1,\ldots,\hat{\xi}_n)$.

P r o o f. We know that

$E^S H_n(\xi_1,\ldots,\xi_n) = P_n(\hat{\xi}_1,\ldots,\hat{\xi}_n)$ and

$E^S H_1(\xi_1) = E^S\xi_1 = \hat{\xi}_1$.

The conditional distribution of $(\xi_1,\xi_2,\ldots,\xi_n)$, given $F(S)$, is a Gaussian with the mean vector $(\hat{\xi}_1,\hat{\xi}_2,\ldots,\hat{\xi}_n)$ and covariance matrix $B = \|b_{ij}\|$, where $b_{ij} = E^S(\xi_i-\hat{\xi}_i)(\xi_j-\hat{\xi}_j)$ are constant, because $\xi_k-\hat{\xi}_k$ is independent of $F(S)$. Let $\phi(x_1-\hat{\xi}_1, x_2-\hat{\xi}_2,\ldots x_n-\hat{\xi}_n\,|\,F(S))$ be the conditional density of $(\xi_1,\xi_2,\ldots,\xi_n)$. So

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$$E^S H_n(\xi_1,\dots,\xi_n) = \int_{R^n} H_n(x_1,\dots,x_n)\phi(x_1-\hat{\xi}_1,\dots,x_n-\hat{\xi}_n \mid F(S))dx_1 \dots dx_k \dots dx_n$$

and

$$\frac{\partial}{\partial \xi_k} E^S H_n = \int_{R^n} \frac{\partial}{\partial \xi_k} H_n(y_1+\hat{\xi}_1,\dots,y_n+\hat{\xi}_n)\phi(y_1,\dots,y_n \mid F(S))dy_1 \dots$$

$$\dots dy_k \dots dy_n = \int_{R^n} H_{n-1}(y_1+\hat{\xi}_1,\dots,y_{k-1}+\hat{\xi}_{k-1},y_{k+1}+\hat{\xi}_{k+1},$$

$$\dots,y_n+\hat{\xi}_n)\phi(y_1,\dots,y_n \mid F(S))dy_1 \dots dy_n =$$

$$= \int_{R_{n-1}} H_{n-1}(x_1,\dots,x_{k-1},x_{k+1},\dots,x_n)\phi(x_1-\hat{\xi}_1,\dots,x_{k-1}-\hat{\xi}_{k-1},x_{k+1}-$$

$$- \hat{\xi}_{k+1},\dots,x_n-\hat{\xi}_n \mid F(S))dx_1 \dots dx_{k-1}dx_{k+1} \dots dx_n = E^S H_{n-1}(\xi_1,\dots$$

$$\dots,\xi_{k-1},\xi_{k+1},\dots,\xi_n) .$$

It follows that the polynomials $E^S H_n(\xi_1,\dots,\xi_n) = P_n(\hat{\xi}_1,\dots,\hat{\xi}_n)$ satisfy the partial differential equation

$$\frac{\partial}{\partial \hat{\xi}_k} P_n(\hat{\xi}_1,\dots,\hat{\xi}_n) = P_{n-1}(\hat{\xi}_1,\dots,\hat{\xi}_{k-1},\hat{\xi}_{k+1},\dots,\hat{\xi}_n), \quad n \geq 2,$$

and since

$$EP_n(\hat{\xi}_1,\dots,\hat{\xi}_n) = EE^S H_n(\xi_1,\dots,\xi_n) = 0,$$

from Theorem 2. we have

$$E^S H_n(\xi_1,\dots,\xi_n) = P_n(\hat{\xi}_1,\dots,\hat{\xi}_n) = H_n(\hat{\xi}_1,\dots,\hat{\xi}_n) .$$

REMARK 2. In |4| it is shown that for $\eta \in H_p(T)$, $E^S \eta \in H_p(S)$. The same result follows immediately from Theorem 5.

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REZIME

O ERMITSKIM POLINOMIMA GAUSOVOG SLUČAJNOG PROCESA

U radu se daje eksplicitan izraz za Ermitske polinome stepena n, n = 1,2,..., i ispituju se neke osobine ovih polinoma (Teoreme 3., 4., 5.) .

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ON SPECTRAL TYPE OF NONLINEAR AND NONANTICIPATIVE TRANSFORMATION OF THE WIENER PROCESS

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ABSTRACT

Let $H(W)$ be a Hilbert space of the square integrable functions of the Wiener proces $W(T), t \geq 0$. It is shown that there exists a process $\eta(t) = T\{W(u),\ 0 \leq u \leq t\},\ t \geq 0$ which has any given spectral type.

Let $\{E_t,\ t \geq t_o\}$ be a resolution of the identity in a separable Hilbert space H. According to the well-known Stone theorem the spectral type

$$R_1(t) \geq R_2(t) \geq \ldots \geq R_N(t), \quad t \geq t_o \tag{1}$$

(the spectral multiplicity N may be $+\infty$) of $\{E_t\}$ is the complete system of the unitary invariants of $\{E_t\}$.

The Stone theorem is applied to a time-domain analysis of stochastic processes in a classical paper [1] in the following way: Let the second order process $\{\xi(t),\ t \geq t_o\}$ be continuous and purely nondeterministic. Let $H^{(1)}(\xi)\ (H^{(1)}_\xi(t))$ be a Hilbert space - the linear closure over $\xi(t),\ t \geq t_o$ $(\xi(u),\ t_o \leq u \leq t)$. The scalar product is $(\xi_1, \xi_2) = E\xi_1\xi_2, \xi_i \in H^{(1)}(\xi)$, E is the expectation. So the family $\{P_t,\ t \geq t_o\}$ of the projection operators P_t onto $H^{(1)}_t(\xi)$ is the resolution of the identity.

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Then there exist N and the mutually orthogonal wide sense martingals $\{\zeta_n(t),\ t \geq t_o\}$, $n = \overline{1,N}$ such that

(2) 1. $H_t^{(1)}(\xi) = \sum_{n=1}^{N} \oplus H_t^{(1)}(\zeta_n),\quad t \geq t_o$,

2. The measure dF_{ζ_n} with the distribution function $F_{\zeta_n}(t) = \| \zeta_n(t) \|^2$, $t \geq t_o$, belongs to the class of the equivalent measure $R_n^{\xi}(t)$. The spectral type of $\{\xi(t)\}$

$$R_1^{\xi}(t) \geq R_2^{\xi}(t) \geq \ldots \geq R_N^{\xi}(t),\quad t \geq t_o$$

is the spectral type of $\{P_t\}$.

The main result of |1| is that for the arbitrary spectral type (1) there exists a continuous process $\{\xi(t),\ t \geq t_o\}$ with this spectral type.

Let $\{W(t),\ t \geq t_o\}$ be a Wiener process and $\xi(t) = L\{W(u), 0 \leq u \leq t\}$ be a linear nonanticipative transformation of $\{W(t)\}$ i.e. the process

$$\xi(t) = \int_0^t g(t,u)\,dW(u),\quad t \geq 0,\quad g \in L_2(dt).$$

In |8| it is noted that there exists L such that the process has any spectral type (1) for $t \geq \varepsilon$ where $\varepsilon > 0$ is arbitrary but fixed. The idea of the proof of this statement in |4| is to in space $H_{\varepsilon}^{(1)}(W)$ the mutually orthogonal wide-sense martingals $\{\zeta_n(t),\ t \geq t_o\}$, $n = \overline{1,N}$, such that (2) holds. We do not see the way to remove the restriction $t \geq \varepsilon > 0$.

Now let $H(W)$ ($H_t(W)$) be Hilbert space of all random variables η, $E\eta^2 < +\infty$, $E\eta = 0$, measurable with respect to the σ-algebra $F(F_t)$ generated by $\{W(u);\ u \geq 0\}$ ($\{W(u),\ 0 \leq u \leq t\}$). It is well-known that, $\{W(t)\}$ being a Gaussian process:

1. the conditional expectation $E(.|F_t)$ is the projection operator E_t onto $H_t(W)$,

2. the subspace $H_t^{(1)}(W)$, of $H_t(W)$ reduces $\{E_t\}$ to $\{P_t\}$.

In this way, $\{E_t,\ t \geq 0\}$ is the resolution of the identity and it has its spectral type. The "part" $\{P_t\}$ of $\{E_t\}$ in $H^{(1)}(W)$

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has the spectral type R_1 to which all measures equivalent to the measure $d\|W(t)\|^2 = dt$ belong.

In this paper we shall show that it is possible to remove the restriction $t \geq \varepsilon > 0$ in |8| if we extend the space $H^{(1)}(W)$ to $H(W)$. We find the justification for this extension in the fact that $H^{(1)}(W)$ reduces $\{E_t\}$ to $\{P_t\}$.

THEOREM 1. *There exists a continuous second order process* $\{\eta(t), t \geq 0\}$ *in* $H(W)$ *such that:*

1. $\eta(t)$ *is the nonanticipative transformation of* $\{W(u),\ 0 \leq u \leq t\}$ *i.e.* $\eta(t) = T\{W(u),\ 0 \leq u \leq t\} \in H_t(W)$ *and*

2. *the spectral type of* $\{\eta(t)\}$ *is any given spectral type (1).*

P r o o f. For the sake of simplicity, it is not the restriction if we suppose that the spectral types $R_n(t)$, $n = \overline{1,N}$ are equivalent to an ordinary Lebesgue measure dt.

The existence of the process $\{\eta(t)\}$ essentially depends on the existence of the mutually orthogonal martingals $\{\zeta_n(t), t \geq 0\}$, $n = \overline{1,N}$ for which $\zeta_n(t) \in H_t(W)$ and $dF_{\zeta_n}(t) = f_n(t)dt$, $f_n(t) > 0$, a.e. . For the construction of the martingals $\{\zeta_n(t)\}$, $n = \overline{1,N}$, we shall use the Hermite polynomials of the Gaussian process. The explicit formulae for these polynominals and some of their properties are given in |5|. We consider the Hermite polynomial of degree n of the random variable $W(t)$:

$$(3)\quad H_n(W(t)) = W^n(t) + \sum_{k=1}^{\left[\frac{n}{2}\right]} (-1)^k (2k-1)!! \binom{n}{2k} t^k W^{n-2k}(t)$$

It is proved in |5| that $E_s H_n(W(t)) = H_n(W(s))$, $s < t$. It follows from this relation that $\{H_n(W(t)), t \geq 0\}$ is the martingal and that the subspace $H^{(1)}(H_n)$ reduces $\{E_s\}$. As $H_m(W(t))$ and $H_n(W(s))$ are orthogonal for $m \neq n$ we conclude that the space $H = \sum_{n=1}^{N} \oplus H^{(1)}(H_n)$ is the subspace of $H(W)$ and that

$$(4)\quad E_s H = \sum_{n=1}^{N} \oplus H_s^{(1)}(H_n),\ s \geq 0 .$$

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Now we can find, following the construction in |1|, the continuous process $\{\eta(t),\ t \geq 0\}$ for which $H_s^{(1)}(\eta) = E_s I'$, $s \geq 0$. There remains to show that the spectral type of $\{\eta(t)\}$ is the given type (1). From

$$H_s^{(1)}(\eta) = \sum_{n=1}^{N} \oplus H_s^{(1)}(H_n),\ s \geq 0,$$

it follows that the spectral multiplicity of $\{\eta(t)\}$ is N. We have $F_{H_n}(t) = EH_n^2(W(t)) = n!t^n$, $t \geq 0$. It means that the spectral type of $\{H_n(W(t)),\ t \geq 0\}$, is equivalent to dt.

COROLLARY 1. *There exosts a second order continuous* N*-ple Markov process (in the sense of* |2|) $\{\eta(t),\ t \geq 0\}$ *such that:*

1. $\eta(t)$ *is the random variable measurable with respect to the* σ*-algebra generated by* $W(t)$ *i.e.* $\eta(t) = T_t(W(t))$ *for each* $t \geq 0$ *and*

2. $\{\eta(t)\}$ *has the finite spectral multiplicity* N .

We may take

$$\eta(t) = H_1(W(t)) \oplus \phi(t)H_2(W(t)) \oplus \ldots \oplus \phi^{N-1}(t)H_N(W(t))$$

where ϕ is a continuous function not absolutely continuous in any interval (|3|, |7|, |6|).

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REZIME

O SPEKTRALNOM TIPU NELINEARNE I NEANTICIPIRAJUĆE TRANSFORMACIJE VINEROVOG PROCESA

Neka je $H(W)$ Hilbertov prostor kvadrat integrabilnih funkcionela Vinerovog procesa $W(t)$, $t \geq 0$. U radu se pokazuje da postoji proces

$$\eta(t) = T\{W(u),\ 0 \leq u \leq t\},\ t \geq 0$$

koji ima unapred zadani proizvoljni spektralni tip.

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SOME DIFFERENCE SCHEMES FOR TWO POINT BOUNDARY VALUE PROBLEMS

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ABSTRACT

This paper is concerned with discretizing the boundary value problems in ordinary differential equations. We set up a total of 4 schemes for a boundary value problem $-x'' = f(t,x)$ on $[0,1]$, $R_i x = \gamma_i$ $(i=1,2,)$, with four classes of linear functionals R_i on $C^1[0,1]$ on a nonuniform mesh.

1. INTRODUCTION

We shall consider a boundary value problem of the from

(1) $-x'' = f(t,x), \quad t \in I = [0,1], \quad R_i x = \gamma_i, \quad i=0,1,$

with four classes of linear functionals R_i on $C^1(I)$. We assume that $f \in C(I \times \mathbb{R})$ and $\gamma_i \in \mathbb{R}$, $i=0,1$. Further assumptions will come into the discussion later.

Let $n \in \mathbb{N}$, $n \geq 3$. With $k_j \in \mathbb{R}$, $k_j > 0$, $j = 1,2,\ldots,n$ we define a nonuniform mesh

(2) $I_h = \{t_o = 0, \; t_j = t_{j-1} + hk_j; \; j=1,2,\ldots,n\}$,

where

$$h^{-1} = \sum_{j=1}^{n} k_j .$$

For the numerical solution of problem (1) we form a discrete analogue to (1) with canonical form

(3) $$A_h x = B_h F_h x + r_h [\gamma_o, \gamma_1] \quad \text{in} \quad \mathbb{R}^{I_h},$$

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where $A_h, B_h \in L(\mathbb{R}^{I_h})$ (= set of all linear operators on $\mathbb{R}^{I_h}$) and where $r_h \in L(\mathbb{R}^2, \mathbb{R}^{I_h})$. For any of our schemes F_h is the non-linear mapping to $\mathbb{R}^{I_h}$ into itself which assigns to $x \in \mathbb{R}^{I_h}$ the element $F_h x \in \mathbb{R}^{I_h}$ whose t-th component is given via

$$(F_h x)(t) = f(t, x(t)) , \quad t \in I_h .$$

Any sequence of discrete problems (3) defines a (finite difference)scheme for the boundary value problem (1).

The i-th equation of (3) reads

$$\sum_{j=o}^{n} A_h(i,j)x_j = \sum_{j=o}^{n} B_h(i,j)F_h x_j + r_h[\gamma_o, \gamma_1](i) .$$

We abbreviate this as

$$(A_h(i,0), \ldots, \underline{A_h(i,i)}, \ldots, A_h(i,n)) =$$
$$(B_h(i,0), \ldots, \underline{B_h(i,i)}, \ldots, B_h(i,n)) + r_h[\gamma_o, \gamma_1](i),$$

where we shall leave out zero entries and where we shall write the common factors of the entries of the respective matrices in front of the parentheses, see |2|.

We form the finite difference schemes for (1) by using the second order approximation of $-x''$ and $R_i x$ from |4|, |5|. Now we shall describe this formula.

Let

$$x \in C^4(I),\ h \in \mathbb{R},\ h > 0,\ \alpha_j \in \mathbb{R}\setminus\{0\}, j=1,2,3,\ \alpha_i \neq \alpha_j$$

if $i \neq j$, $i,j = 1,2,3$, and $t,\ t + h\alpha_j \in I$, $j=1,2,3$. Then

$$(4) \quad -x''(t) = h^{-2}(ax(t+\alpha_1 h) + bx(t) + cx(t+\alpha_2 h) + dx(t+\alpha_3 h)) + O(h^2)$$

$$(5) \quad x'(t) = h^{-1}(\hat{a}x(t+\alpha_1 h) + \hat{b}x(t) + \hat{c}x(t+\alpha_2 h)) + O(h^2)$$

where

$$a = \frac{2(\alpha_2 + \alpha_3)}{\alpha_1(\alpha_1-\alpha_2)(\alpha_1-\alpha_3)}, \qquad b = \frac{-2(\alpha_1+\alpha_2+\alpha_3)}{\alpha_1\ \alpha_2\ \alpha_3},$$

(6)

$$c = \frac{2(\alpha_1 + \alpha_3)}{\alpha_2(\alpha_2-\alpha_1)(\alpha_2-\alpha_3)}, \qquad d = \frac{2(\alpha_1 + \alpha_2)}{\alpha_3(\alpha_3-\alpha_1)(\alpha_3-\alpha_2)},$$

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(7) $\hat{a} = \frac{-\alpha_2}{\alpha_1(\alpha_1-\alpha_2)}$ $\hat{b} = -\frac{\alpha_1+\alpha_2}{\alpha_1\alpha_2}$, $\hat{c} = \frac{-\alpha_1}{\alpha_2(\alpha_2-\alpha_1)}$

All the schemes described in this paper in the case that I_h is a uniform mesh with the step width $h = n^{-1}$, $n \in \mathbb{N}$, are the same as in |2|. Also properties of our matrices A_h, we shall prove by using the methods from |1|,|2|,|8|,|9|.

In |6|,|7| is also considered the fourth order Hermitian approximation of (1) and the second order approximation of

(8) $-x'' + p(t)x = f(x)$, $t \in I$, $R_i x = \gamma_i$, $i=0,1$, , $p \in C(I)$.

Also in |3| are given some difference schemes on a nonuniform mesh (2) for (1) and (8). So, we have 12 different schemes on a nonuniform mesh which, in the special case when the mesh is uniform, are the same as in |2|.

2. FINITE DIFFERENCE SCHEMES

In this section we describe the schemes which we are going to be discussed. We separate four cases of boundary constraints (as in |2|):

I Dirichlet conditions

$$Rx = x(0) , \quad R_1 x = x(1)$$

II The uncoupled boundary conditions involving derivatives at both endpoints

$$R_o x = g_o\, x(0) - x'(0), \quad R_1 x = g_1 x(1) + x'(1) ,$$

$$g_o \geq 0, \quad g_1 \geq 0, \quad g_o + g_1 > 0 .$$

III The uncoupled boundary conditions involving a derivative at one endpoint

$$R_o x = x(0), \quad R_1 x = g_1 x(1) + x'(1), \quad g_1 \geq 0 ,$$

$$R_o x = g_o x(0) - x'(0), \quad R_1 x = x(1), \quad g_o \geq 0,$$

respectively.

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IV Sturm-Liouville conditions of the form

$$R_o x = gx(0) - g_o x'(0) + g_1 x'(1), \quad R_1 x = x(0) - x(1),$$
$$g > 0, \quad g_0 > 0, \quad g_1 > 0 .$$

Thus we assume $\gamma_1 = 0$ in (1).

In the following cases I_h, h are as in the introduction.

CASE I. Let

$$1 \le k_i \le k_{i+1}, \quad i=1,2,\dots,n-2 ,$$

(9) $$k_n = \sum_{j=o}^{p_{n-1}} k_{n-1-j} \quad \text{for some } p_{n-1} \in \{0,1,\dots,n-2\} .$$

We define (with $k_{n+1} := k_n$)

(10) $$\alpha_2(i) = k_{i+1}, \ \alpha_3(i) = \alpha_2(i) + k_{i+2}, \ i=1,2,\dots,n-1 ,$$

(11) $$\alpha_1(i) = - \sum_{j=o}^{p_i} k_{i-j} , \ i = 1,\dots,n-1 .$$

Here p_{n-1} is determined in (9) and $p_i \in \{0,1,\dots,i-1\}$, $= 1,2,\dots,n-2$ we determine so that

(12) $$\alpha_1(i) + \alpha_3(i) \ge 0, \quad i=1,2,\dots,n-2 .$$

It is shown in |4| how one can determine p_i. The choice $p_i = 0$, $i=1,2,\dots,n-1$, and $k_{n-1} = k_n$ also possible.

From (9) - (12) follows

$$\alpha_1(i) < 0, \quad 0 < \alpha_2(i) \le 0.5\alpha_3(i),$$

(13) $$\alpha_1(i) + \alpha_3(i) \ge 0, \quad i=1,2,\dots,n-1,$$

(14) $$\alpha_1(n-1) + \alpha_2(n-1) = 0,$$

$$t_i + \alpha_j(i)h \in I_h, \ j=1,2; \quad i=1,2,\dots,n-1,$$
$$t_i + \alpha_3(i)h \in I_{h-1}, \ i=1,2,\dots,n-2.$$

Let a_i, b_i, c_i, d_i be determined by (6), as a,b,c,d respectively, with $\alpha_1(i)$, $\alpha_2(i)$, $\alpha_3(i)$. Then we have the following scheme.

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Scheme I.

$$(\underline{1}) = (\underline{0}) + \gamma_i \quad \text{for} \quad t = 0,1,$$

$$h^{-2}(a_i, \underbrace{0,\dots,0}_{p_i}, \underline{b_i}, c_i, d_i) = (\underline{1}) \quad \text{for} \quad t_i \in I_h \setminus \{0,1,t_{n-1}\},$$

$$h^{-2}(a_{n-1}, \underbrace{0,\dots,0}_{p_{n-1}}, \underline{b_{n-1}}, c_{n-1}) = (\underline{1}) \text{for} \quad t = t_{n-1}.$$

REMARK. The assumption (9) is natural in the case when the solution x of (1) has a boundary layer property at $t=0$. If x has this property at $t=1$ we can use the scheme

$$(\underline{1}) = (\underline{0}) + \gamma_i \quad \text{for} \quad t = 0,1$$

$$h^{-2}(c_1, \underline{b_1}, \underbrace{0,\dots,0}_{p_i}, a_1) = (\underline{1}) \quad \text{for} \quad t = t_1$$

$$h^{-2}(d_i, c_i, \underline{b_i}, \underbrace{0,\dots,0}_{p_i}, a_i) = (\underline{1}) \quad \text{for} \quad t \in I_h \setminus \{0,1,t_1\},$$

with $k_1 = k_2$, $k_i \geq k_{i+1} \geq 1$, $i=2,\dots,n-1$, and $\alpha_1(i) = \alpha_2(i)$, $\alpha_3(i)$, as above.

If the solution of x has boundary layer properties at $t=0$ and $t=1$ we define

$$k_{n+i} = k_{n+1-i}, \ i=1,2,\dots,n, \quad h^{-1} = 2\sum_{j=1}^{n} k_j,$$

$$t_o = 0, \ t_{i+1} = t_i + hk_{i+1}, \ i=0,1,\dots,2n-1$$

$$I_h^1 = \{t_i : i=0,1,\dots,n\}, \ I_h^2 = \{t_i : i=n+1,n+2,\dots,2n\},$$

$$-\alpha_1(n) = \alpha_2(n) = k_n, \quad \alpha_3(n) = k_n + k_{n-1}, \quad p_n = 0.$$

Now we use the scheme

$$(\underline{1}) = (\underline{0}) + \gamma_i \quad \text{for} \quad t = 0,1$$

$$h^{-2}(a_i, \underbrace{0,\dots,0}_{p_i}, b_i, c_i, d_i) = (\underline{1}) \quad \text{for} \quad t_i \in I_h^1 \setminus \{0\},$$

$$h^{-2}(d_{2n-i}, c_{2n-i}, \underline{b_{2n-i}}, \underbrace{0,\dots,0}_{p_i}, a_{2n-i}) = (\underline{1}) \quad \text{for} \quad t_i \in I_h^2 \setminus \{1\},$$

with the coefficients a_i, b_i, c_i, d_i as above.

CASE II. In this case all the assumptions are the same as in case I.

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Scheme II. For $t \in I_h \setminus \{0,1\}$ as in case I, and

$$h^{-1}(-\hat{b}_o+hg_o, -\hat{a}_o, -\hat{c}_o) = (\underline{0}) + \gamma_o \quad \text{for} \quad t = 0,$$

$$h^{-1}(\hat{c}_n, 0, \ldots, 0, \hat{a}_n, \hat{b}_n+hg_1) = (\underline{0}) + \gamma_1 \quad \text{for} \quad t = 1,$$

where $\hat{a}_o$, $\hat{b}_o$, $\hat{c}_o$, are given by (7) with

$\alpha_1 = -\alpha_1(1) = k_1$, $\alpha_2 = \alpha_2(1) - \alpha_1(1) = k_1 + k_2$, and

$\hat{a}_n, \hat{b}_n, \hat{c}_n$ with $\alpha_1 = -\alpha_2(n-1) = -k_n$, $\alpha_2 = -\alpha_2(n-1) + \alpha_1(n-1) = -2k_n$. So we have

$$(15) \quad \hat{a}_o = \frac{k_1+k_2}{k_1k_2}, \quad \hat{b}_o = -\frac{2k_1+k_2}{k_1(k_1+k_2)}, \quad \hat{c}_o = \frac{-k_1}{k_2(k_1+k_2)}$$

$$(16) \quad \hat{a}_n = \frac{-2}{k_n}, \quad \hat{b}_n = \frac{3}{2k_n}, \quad \hat{c}_n = \frac{1}{2k_n}$$

CASE III.

Scheme III

$R_o x = x(0)$: for t=0 as in I and for $t \in I_h \setminus \{0\}$ as in II

$R_1 x = x(1)$: for t=1 as in I and for $t \in I_h \setminus \{1\}$ as in II

CASE IV. Let $n > 3$, $I_h = \{t_o=0, t_j=t_{j-1}+k_j h \quad j=1,2,\ldots,n-1\}$. Let k_i, i=1,2,...,n, be satisfied (9) by $p_{n-1} = 0$. The other assumptions are as above.

Scheme IV. For $t \in I_h \setminus \{0, t_{n-1}, t_{n-2}\}$ as in I,

$$h^{-1}(\underline{-\hat{b}_o g_o + \hat{b}_n g_1 + gh}, \; -\hat{a}_o g_o, \; -\hat{c}_o g_o, 0, \ldots, 0, \hat{c}_n g_1, \hat{a}_n g_1) = (\underline{0}) + \gamma_o \quad \text{for } t=0,$$

$$h^{-2}(d_{n-2}, 0, \ldots, a_{n-2}, \underbrace{0, \ldots, 0}_{p_{n-2}}, \underline{b_{n-2}}, c_{n-2}) = (\underline{1})$$

for $t=t_{n-1}$ and $p_{n-2} < n-3$,

$$h^{-2}(d_{n-2}+a_{n-2}, 0, \ldots, 0, \underline{b_{n-2}}, c_{n-2}) = (\underline{1}) \quad \text{for } t=t_{n-2} \text{ and}$$

$$p_{n-2} = n-3,$$

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$$h^{-2}(c_{n-1},0,\ldots,0,a_{n-1},\underline{b_{n-1}}) = (\underline{1}) \quad \text{for } t = t_{n-1} .$$

3. PROPERTIES OF THE SCHEMES I - IV

We begin with some properties of the coefficients of schemes I-IV.

From (6) and (13) it follows that

$$a_i < 0, \quad b_i > 0, \quad c_i \underline{<} 0 ,$$

$$(17) \qquad d\begin{cases} \underline{<} 0 & \text{for } \alpha_1(i)+\alpha_2(i) \underline{<} 0, \\ = 0 & \text{for } \alpha_1(i)+\alpha_2(i) > 0 , \end{cases}$$

and from (7)

$$(0 < \alpha_1(i) < \alpha_2(i)) \Longrightarrow (\hat{a}_i > 0, \hat{b}_i < 0, \hat{c}_i < 0),$$

$$(18) \quad (\alpha_1(i) < 0, \alpha_1(i)+\alpha_2(i) \underline{>} 0) \Longrightarrow \hat{a}_i < 0, \hat{b}_i \underline{>} 0, \hat{c}_i > 0) ,$$

$$(\alpha_1(i) < \alpha_2(i) < 0) \Longrightarrow (\hat{a}_i > 0, \hat{b}_i > 0, \hat{c}_i < 0) .$$

In case $\alpha_1(i) + \alpha_2(i) = 0$, we have

$$(19) \qquad \hat{b}_i = 0, \quad \hat{a}_i = -\hat{c}_i = -(2\alpha_2(i))^{-1}$$
$$d_i = 0, \quad a_i = c_i = -0.5b_i = -\alpha_2(i)^{-2} ,$$

and for $\alpha_1(i) + \alpha_3(i) = 0$

$$(20) \qquad c_i = 0, \quad a_i = d_i = -0.5b_i = -\alpha_3(i)^{-2}$$

All our schemes are based on scheme I, and it serves as a basis for our study. The matrix A_h from scheme I is

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$$A_h = h^{-2}\begin{bmatrix} h^2 & & & & & & \\ a_{10} & b_1 & c_1 & d_1 & & & \\ a_{21} & a_{20} & b_2 & c_2 & d_2 & & \\ \vdots & \ddots & \ddots & \ddots & \ddots & \ddots & \\ a_{n-2,n-3} & a_{n-2,n-4}\cdots & a_{n-2,0} & b_{n-2} & c_{n-2} & d_{n-2} & \\ a_{n-1,n-2} & a_{n-1,n-3}\cdots & & a_{n-1,0} & b_{n-1} & c_{n-1} & \\ & & & & & h^2 & \end{bmatrix}$$

where for $i=1,2,\ldots,n-1$, $j=0,1,2,\ldots,i-1$,

$$a_{ij} = \begin{cases} a_i & \text{for} \quad j = p_i \\ 0 & \text{for} \quad j \neq p_i \end{cases}$$

Only the coefficients $A_h(i,i+2) = d_i$, $i=1,2,\ldots,n-2$ can be positive. If we define

$$\tau_d^+ = \{i: \alpha_1(i) + \alpha_2(i) > 0, \quad i=1,2,\ldots,n-2\}$$

we have

$$d_i > 0 \quad \text{if} \quad i \in \tau_d^+ .$$

Now from (17) if follows (with $d_{n-1} = 0$) that

$$a_i < 0, \quad c_i \begin{cases} < 0 & \text{if} \quad d_i \geq 0 , \\ \leq & \text{if} \quad d_i > 0 . \end{cases}$$

We are using notations from |1|, |2| (see also |3|) and define $A_{h,i}$, $i=1,2,3,4$, as a matrix from scheme I, II, III or IV.

THEOREM 1. *The matrix* $A_{h,i}$, $i=1,2,3,4$ *is inverse monotone if* $\tau_d^+ \neq \emptyset$ *and if for each* $i \in \tau_d^+$

(21) $$\alpha_1(i+1) \geq z_1$$

where z_1 *is smaller solution of the equation*

$$z^2 + \alpha_3(i+1)z + \alpha_2(i+1)\Delta = 0$$

with

$$\Delta^{-1} = \frac{-1}{\alpha_3(i+1)+\alpha_2(i+1)} + \frac{\alpha_3(i+1)\alpha_3(i)(\alpha_3^2(i) - \alpha_1^2(i))}{4\,\alpha_2(i)(\alpha_2^2(i)-\alpha_1^2(i))(\alpha_3^2(i+1)-\alpha_2^{2(i+1)})}$$

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If $\tau_d^+ = \emptyset$ *then* $A_{h,1}$ *is an* M-*matrix.*

P r o o f. In case $A_{h,1}$ the proof is given in |5|. Now we are going to prove that $A := h^2 A_{h,2}$ is inverse monotone i.e. that $A_{h,2}$ is inverse monotone. In other cases the proof is analogous. We are using the notations and theorems from |1| |2|, |3|. Let $A_d = \mathrm{diag}(h^2, b_1, \ldots, b_{n-1}, h^2)$,

$$\tilde{d}_i = \begin{cases} d_i & \text{for } i \in \tau_d^+ \\ 0 & \text{for } i \notin \tau_d^- \end{cases} \qquad i=1,2,\ldots,n-2$$

$$A_o^+ = \begin{bmatrix} 0 & 0 & 0 & \ldots & 0 & -\hat{c}_o h \\ & 0 & 0 & \tilde{d}_1 & & \\ & & 0 & 0 & \tilde{d}_2 & \\ & & & \ddots & \ddots & \ddots \\ & & & & 0 & 0 & \tilde{d}_{n-2} \\ & & & & 0 & 0 & 0 \\ \hat{c}_{n,n-1} & \hat{c}_{n,n-2} & \ldots & \hat{c}_{n0} & 0 & 0 \end{bmatrix}$$

$$B = \begin{bmatrix} 0 & -\hat{a}_o h & & & & & \\ 0.5a_1 & 0 & g_1 c_1 & d_1 - \tilde{d}_1 & & & \\ a_{21} & a_{20} & 0 & 0.5c_2 & d_2 - \tilde{d}_2 & & \\ \vdots & \vdots & \vdots & \vdots & \vdots & & \\ a_{n-2,n-3} & & & a_{n-2,0} & 0 & 0.5c_{n-2} & d_{n-2} - \tilde{d}_{n-2} \\ qa_{n-1,n-2} & & & qa_{n-1,1} & qa_{n-1,0} & 0 & 0.5c_{n-1} \\ 0 & & & 0 & 0 & \hat{a}_n h & 0 \end{bmatrix}$$

where for some $q \in (0, 0.5)$

$$g_1 = \begin{cases} 0.5 & \text{for } k_1 < k_2, \\ q & \text{for } k_1 = k_2 \end{cases}, \qquad a_{ij} = \begin{cases} 0.5a_i & \text{for } j = p_i \\ 0 & \text{for } j \neq p_i \end{cases} \quad \begin{matrix} i=1,2,\ldots,n-1, \\ j=0,1,\ldots,i-1, \end{matrix}$$

$$\hat{c}_{nj} = \begin{cases} 0 & \text{for } j \neq p_{n-1} \\ \hat{c}_n h & \text{for } j = p_{n-1} \end{cases} \qquad j=0,1,\ldots,n-1$$

Let $A^- = A - A_d - A_o^+$, $C = A^- - B$, then $B \leq 0$ and $C \leq 0$. The ML-criterion yields that A is inverse monotone if

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(i) $A \leq ML$, where

$$M = A_d + B\,, \quad L = E + A_d^{-1} C\,,$$

(ii) M is an M-matrix and $L_o \leq 0$,

(iii) there exists $e > 0$ such that $Ae \geq 0$ and M or L connects $\tau^o(Ae)$ with $\tau^+(Ae)$.

The estimate $A \leq ML$ holds if and only if

$$4d_i b_{i+1} \leq c_i c_{i+1}\,, \quad \text{for } i \in \tau_d^+\,, \tag{22}$$

$$-\hat{c}_o h b_1 \leq -(1-q_1)\hat{a}_o h c_1\,,$$

$$\hat{c}_n h b_{n-1} \leq (1-q) a_{n-1} \hat{a}_n h$$

The estimate (22) is proved in [3], and the following two can be easily proved.

With $\delta = (1, 1, \ldots, 1, 1)$ we have

$$\tau^o(M\delta) = \begin{cases} \{0\} & \text{if } g_o = 0,\ g_1 > 0\,, \\ \emptyset & \text{if } g_o > 0,\ g_1 > 0\,, \\ \{n\} & \text{if } g_o > 0,\ g_1 = 0. \end{cases}$$

Since $M_o = B \leq 0$ and M connects $\tau^o(M\delta)$ with $\tau^+(M\delta) = \{0,1,\ldots,n\} \setminus \tau^o(M\delta)$ from the M-criterion it follows that M is an M-matrix. Now we see that (ii) is satisfied since $L_o = A_d^{-1} C \leq 0$.

In case II we have $g_o \geq 0$, $g_1 \geq 0$, $g_o + g_1 > 0$, and

$$\tau^+(A\delta) = \begin{cases} \{0\} & \text{if } g_o > 0,\ g_1 = 0, \\ \{0,n\} & \text{if } g_o > 0,\ g_1 > 0\,, \\ \{n\} & \text{if } g_o = 0,\ g_1 > 0\,. \end{cases}$$

In first two cases the matrix M connects the set $\tau^o(A\delta) = \{0,1,\ldots,n\} \setminus \tau^+(A\delta)$ with $\tau^+(A\delta)$ so that for $i_o \in \tau^o(A\delta)$ we define

$$i_{j+1} = i_j - 1 - p_{i_j}\,, \quad j = 0,1,\ldots,r-1,$$

$$i_r = 0,$$

where $i_o = n \Rightarrow p_{i_o} = 0$.

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In the case that $g_o = 0$, $g_1 > 0$ the matrix M connects $\tau^o(A\delta) = \{0,1,\ldots,n-1\}$ with $\tau^+(A\delta) = \{n\}$ so that for $i_o \in \tau^o(A\delta)$ we form

$$i_{j+1} = i_j + 1, \quad j=0,1,\ldots,r-1$$

$$i_r = n.$$

Now from the ML-criterion it follows that A is an inverse monotone matrix.

So, we have $A^{-1}_{h,i} \geq 0$, $i=1,2,3,4$ and $B_h = \mathrm{diag}(0,1,\ldots,1,0)$ in each case I-IV. Before giving Theorem 2 we shall introduce some notations. Let

$$Q_1 = b_1 - c_1(1-z)(1+k_2/k_1)^2,$$

$$z \in (0,0.5) \quad \text{if} \quad k_1 = k_2,$$

$$z = 0.5 \quad \text{if} \quad k_1 < k_2,$$

$$Q_i = \begin{cases} \infty & \text{if} \quad i-1 \notin \tau^+_d \\ \dfrac{c_i c_{i-1}}{4d_{i-1}} - b_i, & \text{if} \quad i-1 \in \tau^+_d \end{cases} \qquad i=2,3,\ldots,n-1,$$

$$S = b_{n-1}(1-2z).$$

THEOREM 2. *If for* $i \in \tau d^+$ (21) *is satisfied then there exists the smallest positive eigenvalue* $\lambda_{h,j}$ *for the eigenvalue problem* $A_{h,j}x = \lambda B_h x$, $j=1,2,3,4$, *and the matrices* $A_{h,j} - B_h D_h$, $j=1,2,3,4$ *are inverse-monotone for any diagonal matrix* $D_h = \mathrm{diag}(\mu_o,\mu_1,\ldots,\mu_n)$ *whose diagonal elements are all in* $[-\bar{q}_i h^{-2}, \lambda_{h,j})$, $i=1,2,\ldots,n-1$. *The following table shows a* $\bar{q}_i$ *of this type where* $\bar{q} = \infty$ *means that* $[-h^{-2}\bar{q}, \lambda_{h,j}) = (-\infty, \lambda_{h,j})$.

Scheme	I,III ($R_o x = x(0)$)	II,III, ($R_1 x = x(1)$), IV
$\bar{q}_1$	∞	Q_1
$\bar{q}_i$ $(i=2,3,\ldots,n-2)$	Q_i	Q_i
$\bar{q}_{n-1}$	Q_{n-1}	$\min(S, Q_{n-1})$

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The proof of this Theorem is based on the inverse monotonicity of matrices $A_{h,j}$ and it is analogous to the proof of PO from |2|.

REMARK. If we take in (11) $p_i = 0$, $i=1,2,\dots,n-1$ and

$$k_j = \begin{cases} 1 & \text{for } j=1,2,\dots,p-1 \\ k & \text{for } j=p,\dots,n \end{cases} \qquad 1 < p < n,$$

then we have

$$\bar{q}_i = \begin{cases} \infty & \text{if } i \neq p, \\ \dfrac{3}{2k^2(k^2-1)} & \text{if } i = p. \end{cases}$$

Now, we shall consider the nonlinear system (3) with the assumption:

(23) $\quad \underline{q}(v-w) \leq f(t,v) - f(t,w) \leq \mu(v-w), \quad v,w \in \mathbb{R}, \ w \leq v$

for some $\underline{q}, \mu \in \mathbb{R}$, where

(24) $\qquad -h^{-2}\bar{q} \leq \mu \leq \lambda_h$

λ_h is the smallest eigenvalue to $A_h x = \lambda B_h x$,

$$\bar{q} = \min\{\bar{q}_i : i=1,2,\dots,n-1\} .$$

Finally, we shall let $h > 0$ be so small that

(25) $\qquad -h^{-2}\bar{q} < \rho_h := 0.5(\lambda_h + \underline{q}).$

Let the mapping $T_h = A_h - B_h F_h$ be defined by the schemes of cases I-IV.

THEOREM 3. *Let for* $i \in \tau_d^+$ (21) *be satisfied and let* (23),(24),(25) *be satisfied. Then* T_h^{-1} *is* $(A_h - \mu B_h)^{-1}$ *- bounded and for any diagonal matrix* $D_h = \operatorname{diag}(d_0, d_1, \dots, d_n)$ *with* $d_i \in [-h^{-2}\bar{q}_i, \rho_h)$, $i=1,2,\dots,n-1$, *the parallel chord method*

$$x^0 \in \mathbb{R}^{I_h}, \quad (A_h - B_h D_h)x^{k+1} = B_h(F_h - D_h)x^k + r_h; \quad k \in \mathbb{N}$$

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converges for any initial approximation to the only solution of the system $Tx = r_h$.

We have, using the second order approximation of $-x''$: if $1 \le k_i \le k_{i+1} \le k_o$, $i=1,2,\dots,n-1$, for some fixed $k_o \in \mathbb{R}$, then $h \to 0$ if $n \to \infty$. Now we shall prove that we have the implication

$$(26) \qquad x \in C^4(I) \Longrightarrow \|x_h - x^h\|_\infty = O(h^2),$$

where x is the solution of the continuous problem (1), the vector $x_h \in \mathbb{R}^{I_h}$ stands for the restriction of x to the mech I_h, and x^h is the solution of (3) in each case I-IV.

THEOREM 4. *Let the assumptions of Theorem 3 be satisfied, then (26) is true for each case I-V.*

P r o o f. Let $\bar{\lambda}$ be the smallest positive eigenvalue of the eigenvalue problem

$$-x'' = \lambda x \quad \text{an } I, \quad R_i x = 0, \quad i=0,1 .$$

The discrete eigenvalues λ_h tend to $\bar{\lambda}$ as $h \to 0$ (i.e. as $n \to \infty$). So from (24) it follows that $\mu < \bar{\lambda}$, for h sufficiently small.

Using the technique from |2| we obtain that $\|(A_h - \mu B_h)^{-1}\|_\infty$ is uniformly bounded, i.e. there exists $\sigma \ge 0$ independent of h such that $\|(A_h - B_h)^{-1}\|_\infty \le \sigma$ for h sufficiently small. Hence we have

$$\|x-y\|_\infty \le \sigma \|T_h x - T_h y\|_\infty \quad \text{for} \quad x,y \in \mathbb{R}^{I_h}$$

if h is small enough.

Now with $x = x_h$, $y = x^h$ we have

$$|x_h - x^h| \le (A_h - \mu B_h)^{-1} |T_h x_h - r_h| ,$$

i.e.

$$\|x_h - x^h\|_\infty \le \text{const} \cdot h^2 .$$

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REZIME

NEKE DIFERENCNE ŠEME ZA KONTURNE PROBLEME

U radu se daju 4 diferencne šeme na neekvidistantnoj mreži (2) za numeričko rešavanje problema (1). U ekvidistantnom slučaju one se svode na dobro poznate šeme, videti na primer |2|. Za matrice A_h diskretnih analogona (3) formiranih pomoću ovih šema dokazano je da su inverzno monotone. Na osnovu toga kao i u |2| za ekvidistantan slučaj, dati su uslovi pod ko-

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jima postupak paralelne sečice (teorema 3) za rešavanje diskretnog analogona konvergira i numeričko rešenje teži kontinualnom kada broj tačaka mreže teži beskonačnosti.

U radu su korišćene oznake i teoreme iz |2|, koje se mogu naći i u |3| iz ove knjige Zbornika.

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EXIT CRITERIA FOR SOME ITERATIVE METHODS

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ABSTRACT

In this paper we consider the equation $f(x)=0$ on the interval $D=[a,b]$, for a real - valued function f. We use the iterative method $x_{n+1}=\phi(x_n)$, $n=0,1,\ldots$, with a suitably chosen $\phi(x)$ and $x_o \in D$. We accept x_{n+1} as a sufficiently accurate approximation of the exact solution α of the given equation, if we have $|x_{n+1}-x_n|<\varepsilon$, where $\varepsilon>0$ is the pre-assinged tolerance, and if the stopping inequality $|x_{n+1}-x_n| \geq |x_{n+1}-\alpha|$ is valid. For the special functions ϕ we give sufficient conditions for the stopping inequality. As special cases we obtain both Newton's iterative method and the classical regula falsi method. Moreover, we prove the stopping inequality for $n=0,1,\ldots$, for the class of iterative methods which are generated by inverse interpolation.

INTRODUCTION

In this paper we shall consider some iterative methods for determining the unique solution $\alpha \in D$ of the equation $f(x)=0$, where f is a real - valued function defined on an interval $D=[a,b]$. Most of the iterative methods can be writen in the form

(1) $$x_{n+1} = \phi(x_n) ,$$

for some suitable function ϕ and an initial approximation x_o. Under certain conditions the iteration defined by (1) converges to the solution α of $f(x)=0$, i.e. $\alpha = \lim_{n\to\infty} x_n$. In many

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automated numerical algorithms, the calculations of interates x_n of (1) are stopped when the difference between two successive approximations is less than a pre-assigned tolerance. So, one evaluates $x_1, x_2, \ldots,$ and accepts x_{n+1} as a sufficiently accurate approximation of α when

$$|x_n - x_{n+1}| < \varepsilon , \tag{2}$$

where ε is the pre-assinged tolerance. Such a procedure may be justified in terms of a stopping inequality.

DEFINITION *The inequality*

$$|x_n - x_{n+1}| \geq |x_{n+1} - \alpha|$$

will be referred to as the stopping inequality.

The validity of the stopping inequality is sufficient to insure that the value x_{n+1}, accepted as the final result, by the above exit criterion (2), will be within the tolerance ε, i.e.

$$|x_{n+1} - \alpha| < \varepsilon. \tag{3}$$

We shall give sufficient conditions for the stopping inequality for some iterative methods of (1), for the solution of the equation $f(x) = 0$.

As special cases are obtained Newton´s method and the secant iterative method (the classical regula falsi method). The stopping inequality for Newton´s method is proved in [1].

1. SOME SUFFICIENT CONDITIONS FOR THE STOPPING INEQUALITY

In this section we shall consider the equation $f(x) = 0$ on the interval $D = [a,b]$ and the equation $x = \phi(x)$ which has roots which coincide with those of $f(x) = 0$ in the interval D and no others. First we shall give some notations and assumptions.

Let the function f satisfy the following conditions.

(F1) $f(a) < 0, \quad f(b) > 0, \quad$ (F2) $f'(x) > 0, \quad x \in D.$

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The condition (F1) implies that $f \in C(D)$ has a root $\alpha \in (a,b)$, and (F2) implies that $f \in C(D)$ has only one root in (a,b). Let

$$D^- = [a,b), \quad D^+ = (\alpha,b], \quad D_o^- = [a,\alpha], \quad D_o^+ = [\alpha,b] ,$$

and

$$u(x) = \frac{f(x)}{f'(x)} , \quad x \in D .$$

Then, under the assumptions (F1), (F2) for $f \in C(D)$ it holds

$$f(x) < 0, \quad u(x) < 0, \quad x \in D^-, \quad f(x) > 0, \quad u(x) > 0, \quad x \in D^+ .$$

If $f(x) = 0$ has a solution $\alpha \in D$ and if g is any function such that

$$0 < |g(x)| < \infty, \quad x \in D$$

then α is a solution of $f(x) = 0$ if and only if α is a solution $x = \phi(x)$, where

$$\phi(x) = x - u(x)g(x) . \tag{4}$$

THEOREM 1. *Let $f \in C^2(D)$ satisfy condition (F2) and let $\alpha \in D$ be a solution of $f(x) = 0$. Let ϕ be of the form (4), where*

$$g \in C^1(D), \quad g(\alpha) = 1. \tag{G}$$

Then there exists the interval $D_\rho(\alpha) = \{x \mid |x-\alpha| \leq \rho\} \subset D, \rho > 0$, such that for any $x_o \in D_\rho(\alpha)$ the iterates $x_n = \phi(x_{n-1})$, $n=1,2,\ldots$, converge to α and the stopping inequality is valid for all $n = 0,1,\ldots$

P r o o f. From (4) we obtain

$$\phi'(x) = 1 - g(x) + u(x)\left(\frac{f''(x)}{f'(x)}\, g(x) - g'(x)\right) ,$$

and $\phi \in C^1(D)$, $\phi'(\alpha) = 0$. Thus, there exists $D_\rho(\alpha)$ such that

$$|\phi'(x)| \leq L \leq 0.5, \quad x \in D_\rho(\alpha) ,$$

and

$$0 < g(x), \quad x \in D_\rho(\alpha) .$$

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Now, from

$$|\phi(x)-\alpha| = |\phi(x)-\phi(\alpha)| \leq L|x-\alpha| < \rho, \quad x \in D_\rho(\alpha)$$

we conclude that ϕ is a contraction type mapping of $D_\rho(\alpha)$ into itself. Thus, ϕ has a fixed point in $D_\rho(\alpha)$, and this point is α. It is, also, well known that iterates x_n converge to α and that it,

$$|x_{n+1}-\alpha| \leq \frac{L}{1-L}|x_{n+1}-x_n|, \quad n=0,1,2,\ldots,$$

holds.

Since $L \in [0,0.5]$, from this inequality follows the stopping inequality for all $n=0,1,\ldots$.

REMARK 1. The conditions (G) for $g(x)$ are not too strong. For example, the condition $g(\alpha)=1$ is satisfied for all eight one-point iteration functions ϕ from [2], and for all iteration functions generated by the inverse interpolation, paragraph 3. As a special case for $g(x)=1$, $x \in D$ we have Newton's method, for $g(x)=1+u(x)\cdot\frac{f''(x)}{2f'(x)}$, $x \in D$, Chebyshev's method of degree 3 and for

$$g(x) = 1/(1-u(x)\frac{f''(x)}{2f'(x)}), \quad x \in D, \text{ Halley's method.}$$

THEOREM 2. *Let* $f \in C^2(D)$ *satisfy* (F1), (F2) *and*

(F3) $$f''(x) > 0, \quad x \in D.$$

Let

(F4) $$f'(b) \leq 2f'(a),$$

(5) $$g \in C^1(D_o^+), \quad g(x) \geq 1, \quad x \in D_o^+$$

(6) $$\phi'(x) \geq 0, \quad x \in D_o^+.$$

Then for any $x_o \in D^+$ *the iterates* $x_n = \phi(x_{n-1})$, $n=1,2,\ldots$, *converge to the unique solution* α *of* $f(x)=0$ *in* D, *and the stopping inequality is valid for all* $n=0,1,\ldots$.

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P r o o f. The conditions (F1) and (F2) assure us of the existence and uniqueness of a solution α of the equation $f(x) = 0$. From (6) it follows that for $x_o \in D^+$ we have

$$\alpha < x_n < x_{n-1} \leq b$$

either for all $n=1,2,\ldots,$ or for $n=1,2,\ldots,k$ with some fixed $k \in \mathbb{N}$ and $\alpha = x_{k+i}$, $i=1,2,\ldots$. Thus $x_n \in D_o^+$, $n=1,2,\ldots,$ and we can conclude that the limit of the sequence defined by $x_n = \phi(x_{n-1})$, $n=1,2,\ldots,$ exists and that this limit is a solution of the equation $f(x) = 0$. Thus, $\alpha = \lim_{n\to\infty} x_n$. From $x_{n+1} = \phi(x_n)$, $n=0,1,\ldots,$ we obtain for all $n=0,1,\ldots,$

$$x_{n+1} - \alpha = x_n - \alpha - \frac{f(x_n)-f(\alpha)}{f'(x_n)}\, g(x_n),$$

$$= (x_n-\alpha)\left(1 - \frac{f'(\alpha_n)}{f'(x_n)}\, g(x_n)\right), \quad \alpha_n \in (\alpha, x_n),$$

and

$$x_{n+1} - x_n = (x_n-\alpha)\, \frac{f'(\alpha_n)}{f'(x_n)}\, g(x_n) .$$

Now we have

$$x_{n+1}-\alpha = \left(1 - \frac{f'(x_n)}{f'(\alpha_n)\, g(x_n)}\right)(x_{n+1} - x_n), \quad n=0,1,\ldots ,$$

and the stopping inequality is valid if

$$\left| 1 - \frac{f'(x_n)}{f'(\alpha_n)\, g(x_n)} \right| \leq 1 , \quad n=0,1,\ldots ,$$

i.e., if

$$0 \leq \frac{f'(x_n)}{f'(\alpha_n)\, g(x_n)} \leq 2 , \quad n=0,1,\ldots . \tag{7}$$

By the assumption x_o is in D^+ and so $x_1, x_2, \ldots,$ are also. So we have $g(x_n) \geq 1$, $n=0,1,\ldots$. Now, from (F3) and (7) we conclude that

$$0 < \frac{f'(x_n)}{f'(\alpha_n)\, g(x_n)} \leq \frac{f'(x_n)}{f'(\alpha_n)} < \frac{f'(b)}{f'(a)} \leq 2, \quad n=0,1,\ldots ,$$

which completes the proof.

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REMARK 2. For Newton's method we have $g(x) = 1$, $x \in D$ and $\phi'(x) = u(x)f''(x)/f'(x)$. Under the assumptions $f \in C^2(D)$ and (F1)-(F4) we have $\phi'(x) \geq 0$, $x \in D_o^+$ and Theorem 2 shows that for Newton's method the stopping inequality is valid. This is the result of [1].

THEOREM 3. *Let* $f \in C^2(D)$ *satisfy the conditions* (F1)-(F3) *and let*

$$g \in C^1(D_o^-), \quad g(x) \geq 0.5, \quad x \in D_o^-$$

$$\phi'(x) \geq 0, \quad x \in D_o^-$$

Then for any $x_o \in D^-$ *the iterates* $x_n = \phi(x_{n-1})$, $n = 1, 2, \ldots$, *converge to the unique solution* $\alpha \in D$ *of* $f(x) = 0$ *and the stopping inequality is valid for all* $n = 0, 1, \ldots$.

P r o o f. One can easily show that for $x_o \in D^-$

$$a \leq x_{n-1} < x_n < \alpha$$

holds either for all $n = 1, 2, \ldots$, or for $n = 1, 2, \ldots, k$ with some fixed $k \in \mathbb{N}$ and $x_{k+i} = \alpha$, $i = 1, 2, \ldots$. Thus $x_n \in D_o^-$, $n = 1, 2, \ldots$ and $\alpha = \lim_{n\to\infty} x_n$.

As in the proof of Theorem 2, the stopping inequality is valid if (7) holds, where $\alpha_n \in (x_n, \alpha)$, $n=0,1,\ldots$. From (F3) we have $0 < f'(x_n) < f'(\alpha_n)$, $n=0,1,\ldots$. Then, since $g(x_n) \geq 0.5$, $n=0,1,\ldots$, we obtain

$$0 < \frac{f'(x_n)}{f'(\alpha_n)g(x_n)} < \frac{1}{g(x_n)} \leq 2, \quad n=0,1,\ldots ,$$

which implies the stopping inequality.

REMARK 3. Let $f \in C^2(D)$. Under the assumptions (F1)-(F4) the classical regula falsi method can be written as

$$x_o \in D^-,$$

$$x_{n+1} = x_n - \frac{x_n - b}{f(x_n) - f(b)} f(x_n), \quad n=0,1,\ldots ,$$

with

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$$g(x) = \frac{f'(x)}{f(x)-f(b)} (x-b) , \quad x \in D .$$

This method is of the from $x_{n+1} = \phi(x_n)$, $n=0,1,\ldots$, where $\phi(x) = x-u(x)g(x)$. For some $\beta \in (x,b)$, $x \in D^-$, by the mean value theorem we have $g(x) = f'(x) / f'(\beta)$. Now from (F3), (F4) and $x < \beta$ we conclude

$$g(x) = \frac{f'(x)}{f'(\beta)} > \frac{f'(a)}{f'(b)} \geq 0.5, \quad x \in D_o^- .$$

In this case we have

$$\phi'(x) = 1 + \frac{1}{f(b)-f(x)} (f(x)-f(b) \frac{f'(x)}{f'(\beta)}) > 0, \quad x \in D_o^- .$$

Now from Theorem 3 it follows that for the classical regula falsi method the stopping inequality is valid.

REMARK 4. If we replace (F3) and (F4) by

(F3') $f''(x) < 0, \quad x \in D,$

(F4') $f'(a) < 2f'(b) ,$

and D^+, D_o^+, by D^-, D_o^- Theorems 2 is also valid.

If we replace, in Theorem 3, the condition (F3) by (F3'), and D^-, D_o^- by D^+, D_o^+ Theorem 3 is true.

3. THE ORDER OF ITERATION FUNCTIONS GENERATED BY INVERSE INTERPOLATION

In this section we shall study the iteration functions generated by inverse interpolacion which are given in |2|.

Let $\alpha \in D$ be a solution of $f(x) = 0$. Let f' be non-zero in a neighbourhood D of α and let $f^{(s)}$ be continuous in this neighbourhood. Then f has an inverse F and $F^{(s)}$ is continuous in a neighbourhood of zero. Let Q_s be the polinomial whose first s-1 derivative agree with F at the point $y = f(x)$. Then

$$F(t) = Q_s(t) + \frac{F^{(s)}(z(t))}{s!} (t-y)^s$$

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and

$$Q_s(t) = \sum_{j=0}^{s-1} \frac{F^{(j)}}{j!} (t-y)^j ,$$

where $z(t)$ lies in the interval determined by y and t.
Define

$$E_s = Q_s(0) .$$

Hence

$$E_s = \sum_{j=0}^{s-1} \frac{(-1)^j}{j!} F^{(j)} f^j ,$$

or

$$E_s = x - \sum_{j=1}^{s-1} \frac{(-1)^{j-1} F^{(j)}}{j! \, (F')^j} u^j .$$

With the definition

$$Y_j(x) = \left. \frac{(-1)^{j-1} F^{(j)}(y)}{j! \, (F'(y))^j} \right|_{y=f(x)}$$

we can write

$$E_s = x - \sum_{j=1}^{s-1} Y_j u^j ,$$

and

$$\alpha = E_s + \frac{(-1)^s F^{(s)}(z(0))}{s! (F')^s} u^s .$$

Now we can write $E_s(x)$ in the form (4):

$$E_s(x) = x - u(x) g(x) ,$$

where

$$g(x) = \sum_{j=1}^{s-1} Y_j u^{j-1} . \tag{8}$$

If $x_0 \in D$ is some initial estimate of α, we form the sequence

$$x_{n+1} = E_s(x_n), \quad n=0,1,\ldots . \tag{9}$$

Under certain conditions these iterates converge to α. We shall give a sufficient condition for the stopping inequality for the iterative method (9). First we shall give some assumptions.

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In this section let α be the unique solution of $f(x)=0$ in $D=[a,b]$ and let D^-, D_o^-, D^+, D_o^+ be defined as before. Let $s \in N$, $s \geq 2$.

THEOREM 4. *Let* $f \in C^s(D)$ *satisfy* (F1), (F2), (F4) *and*

(F5) $\quad \operatorname{sgn}(f^{(j)}(x)) = (-1)^j, \quad x \in D, \ j = 2,3,\ldots,s$.

Then for any $x_o \in D^+$ *the iterates* x_n, *defined by* (9), *converge to the unique solution* $\alpha \in D$ *of* $f(x)=0$ *and the stopping inequality is valid for all* $n = 0,1,\ldots$.

P r o o f. From (F2) we have $f'(x) > 0$, $x \in D$, and f has an inverse F. In a neighbourhood of zero $F^{(s)}$ is continuous. In |2| the formula for the derivative of the inverse function is given. We have

$$F^{(j)} = (f')^{-j} \sum (-1)^r (j+r-1)! \prod_{i=2}^{j} \frac{(A_i)^{B_i}}{B_i!}, \quad j = 1,2,\ldots,s ,$$

with the sum taken over all non-negative integers B_i such that

$$\sum_{i=2}^{j} (i-1)B_i = j-1 ,$$

and where

$$r = \sum_{i=2}^{j} B_i, \quad A_j(x) = \frac{f^{(j)}(x)}{j!\,f'(x)} , \quad j = 1,2,\ldots,s .$$

For $j=1$, $B_i = 0$ for all i.

Now (F2) and (F5) imply

$$\operatorname{sgn}(A_j(x)) = \operatorname{sgn}(f^{(j)}(x)) = (-1)^j, \quad j = 2,\ldots,s ,$$

and

$$\operatorname{sgn}\left((-1)^r (j+r-1)! \prod_{i=2}^{j} \frac{(A_i)^{B_i}}{B_i!}\right) = (-1)^r \prod_{i=2}^{j} \operatorname{sgn}(A_i^{B_i}) =$$

$$= (-1)^r \prod_{i=2}^{j} (-1)^{iB_i} .$$

Observe that

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$$(-1)^r \prod_{i=2}^{j} (-1)^{iB_i} = (-1)^r (-1)^{\sum_{i=2}^{j} iB_i} =$$

$$= (-1)^r (-1)^{j-1} (-1)^r = (-1)^{j-1} .$$

Thus

$$\text{(10)} \qquad \operatorname{sgn}(F^{(j)}(f(x))) = (-1)^{j-1}, \quad j = 1,2,\ldots,s .$$

This implies

$$\operatorname{sgn}(Y_j(x)) = (-1)^{j-1} \operatorname{sgn}(F^{(j)}(f(x))) = 1, \quad j=1,2,\ldots,s .$$

From (8) and $F'(f(x)) = (f'(x))^{-1}$ we have $Y_1 = 1$, $g \in C^1(D)$ and

$$\text{(11)} \qquad g(x) = 1 + \sum_{j=2}^{s-1} Y_j u^{j-1} \geq 1, \quad x \in D_o^+ .$$

In |2| it is proved that

$$E_{s+1} = E_s - \frac{u}{s} E_s' .$$

Thus

$$E_s' = \frac{(-1)^{s-1} F^{(s)}}{(s-1)!\,(F')^s} u^{s-1} .$$

Now for $x \in D_o^+$ $u^{s-1}(x) \geq 0$ holds and

$$\operatorname{sgn}(E_s'(x)) = (-1)^{s-1} \operatorname{sgn}(F^{(s)}(f(x))) = 1, \quad x \in D ,$$

i.e. $E_s'(x) \geq 0$, $x \in D_o^+$.

From (F5) we have $f''(x) > 0$, $x \in D$ and from (11) follows (5). Now we can apply Theorem 2.

REMARK 5. For $s = 2$ the iterates x_n, defined by (9), are Newton's iterates, and for $s = 3$ Chebyshev's.

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REZIME

IZLAZNI KRITERIJUM ZA NEKE ITERATIVNE METODE

U radu se posmatra rešavanje jednačine $f(x) = 0$ u intervalu $D = [a,b]$, pri čemu je f realna funkcija realne promenljive, itarativnim postupkom $x_{n+1} = \phi(x_n)$, $n=0,1,\ldots$, sa pogodno izabranim $\phi(x)$ i $x_o \in D$. Kao dobra aproksimacija tačnog rešenja α date jednačine uzima se x_{n+1} ako važi $|x_{n+1}-x_n|<\varepsilon$ sa unapred izabranim $\varepsilon > 0$ i ako važi nejednačina zaustavljanja

(1) $$|x_{n+1} - x_n| \geq |x_{n+1} - \alpha| \ .$$

Za posebno izabrane funkcije ϕ dati su dovoljni uslovi za nejednačinu (1), a kao specijalne slučajeve dobijamo Njutnov iterativni postupak i klasičan postupak regula falsi. Takodje je za klasu iterativnih postupaka generisanih inverznom interpolacijom dokazana nejednačina (1) za $n = 0,1,\ldots$.

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STABILITY AND MONOTONICITY PROPERTIES OF STIFF QUASILINEAR BOUNDARY PROBLEMS

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ABSTRACT

The paper is concerned with nonlinear two-point boundary value problems of the singular perturbation type. Besides giving existence and uniqueness statements we investigate two kinds of stability properties of the boundary problems, namely the stability with respect to time-evolution and the stability with respect to small perturbations. Both kinds of stability are essentially obtained by inverse-monotonicity properties of classes of linear boundary value problems.

1. INTRODUCTION

This paper is concerned with boundary value problems (P_ε) of the form

$$T_\varepsilon u \equiv -\varepsilon u'' + \alpha(x,u)u' + \beta(x,u) = 0, \quad x \in [0,1] ,$$

$$(P_\varepsilon) \qquad Ru \equiv (u(0), u(1)) = (A,B),$$

where $\varepsilon > 0$ is small. Among others we shall give conditions implying the existence of a unique solution u of (P_ε) for $0 < \varepsilon \leq \varepsilon_0$. If (P_ε) is treated numerically, let us say with a difference method on a grid $G = \{x_0, \ldots, x_n\}$, then the operator equation (P_ε) is replaced by a finite dimensional system of equations

$$T_{\varepsilon,G} u = \gamma_G, \quad u \in \mathbb{R}^G ,$$

with a solution $u_{\varepsilon,G} \in \mathbb{R}^G$, let us say. The truncation error is

$$\tau_{\varepsilon,G} = T_{\varepsilon,G} u_\varepsilon|_G - T_{\varepsilon,G} u_{\varepsilon,G} ,$$

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where $u_{\varepsilon|G}$ denotes the restriction of u_ε to the grid G. Thus it is the question of the stability of $T_{\varepsilon,G}$ to estimate the error $u_{\varepsilon|G} - u_{\varepsilon,G}$ by $\tau_{\varepsilon,G}$. The stability of $T_{\varepsilon,G}$ can be stated in the form of a stability inequality |5|.

$$\|u-v\| \leq c_\varepsilon \| T_{\varepsilon,G}u - T_{\varepsilon,G}v\|' \quad \text{for all } u,v \in \mathbb{R}^G ,$$

where $\|\ \|$, $\|\ \|'$ are suitable norms and c_ε is the stability constant. It cannot be expected that $T_{\varepsilon,G}$ has better stability properties than T_ε itself, and the stability properties of T_ε can be very bad for small ε. It follows from |1,19| that even in some linear cases of (P_ε) the stability constants for T_ε with respect to the maximum norm grow like $e^{\sigma/\varepsilon}$ with $\sigma > 0$ for $\varepsilon \to 0^+$. It is a purpose of this paper to separate cases with better stability behaviour: we shall give conditions under which the stability constants are $O(1)$ or $O(\varepsilon^{-1/2})$ or $O(\varepsilon^{-1})$ for $\varepsilon \to 0^+$. These cases should be treatable by suitable numerical methods for fairly small values of ε (cf. |2,6,11,13-15,20-24; 29-31|) though standard methods may fail.

Throughout the paper we shall work with monotonicity methods - with respect to the natural ordering, and there are some relations of our techniques to the method of differential inequalities as used in |9,16,17|. But the intention of the present paper is different. In |16,17| the essential assumption is that the reduced differential equation ($\varepsilon = 0$) has one or more solutions with certain properties, and then it is investigated whether there is a solution of the ε-dependent problem in a neighbourhood of these solutions. These valuable techniques might lead to a priori estimates for u_ε. In the present paper we shall investigate the inverse of (T_ε, R) without referring to a reduced problem, and we are motivated by the stability question as mentioned above. Actually, if a priori estimates for u_ε are known - for example obtained by the methods of |9,16,17|, then it is more likely that our conditions can be fulfilled by altering the coefficients $\alpha(x,u)$ and $\beta(x,u)$ outside the a priori domain. In this way the two techniques might

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be used together. Nevertheless, we always take global assumptions for $\alpha(x,u)$ and $\beta(x,u)$, and if these are met, then an a priori analysis is not necessary.

The solutions u_ε of (P_ε) describe the stationary states of the evolution equation

$$
\begin{aligned}
&u_t - \varepsilon u_{xx} + \alpha(x,u)u_x + \beta(x,u) = 0, \quad x \in [0,1], \ t > 0, \\
&u(0,t) = A, \quad u(1,t) = B, \quad t > 0 \qquad (1) \\
&u(x,0) = u_0(x), \quad x \in [0,1],
\end{aligned}
$$

which arises among others in convective-diffusion type flow problems (cf. |10|). Under our conditions (to be described below) the solution u_ε of (P_ε) is always a stable state of the evolution equation. Concerning (1) it is of great interest whether T_ε is monotone in the sense of Minty |25|. However, it turns out that T_ε is never Minty-monotone if α actually depends on u and ε is sufficiently small. This is one reason for us to investigate T_ε with monotonicity concepts as induced by the natural ordering.

Conceptually this paper owes to |6| where the same concepts of a stability inequality and of monotonicity are used, and where stability and inverse monotonicity for T_ε and for discrete analogues was proved under the special assumption $\alpha(x,u) = \alpha(x) \geq \alpha_0 > 0$. I thank Prof.DrE.Bohl for his encouraging interest and for discussions about these problems.

2. PRELIMINARIES AND STATEMENT OF THE MAIN RESULTS

The problem (P_ε) is considered under the assumptions

$$A, B, \varepsilon \in \mathbb{R}, \ \varepsilon > 0, \ \alpha, \alpha_u, \beta, \beta_u \in C([0,1] \times \mathbb{R})$$

where $\alpha_u = \frac{\partial \alpha}{\partial u}$ etc. In cases II and IV below also $\alpha_x \in C([0,1] \times \mathbb{R})$ is required. R always denotes the boundary operator

$$Ru = (u(0), u(1))$$

and we use the pointwise ordering, i.e.

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$$u \leq v \iff u(x) \leq v(x) \quad \text{for all} \quad x \in [0,1],$$
$$u < v \iff u(x) < v(x) \quad \text{for all} \quad x \in [0,1],$$
$$Ru \leq Rv \iff u(0) \leq v(0) \quad \text{and} \quad u(1) \leq v(1)$$

where $u,v \in C[0,1]$. For any $u \in C^2[0,1]$ the linearization of T_ε at u is

$$(2) \quad (DT_\varepsilon u)w = -\varepsilon w'' + \alpha(x,u)w' + \{\alpha_u(x,u)u' + \beta_u(x,u)\}w, w \in C^2[0,1].$$

Concerning the evolution equation (1) the following definition is in common use.

DEFINITION 1. *A solution u_ε of (P_ε) is called stable, if the smallest eigenvalue of the eigenvalue problem*

$$(DT_\varepsilon u_\varepsilon)w = \lambda w, \quad Rw = (0,0)$$

is positive.

Because of the following Lemma the stability of u_ε in the sense of Definition 1 is equivalent to the inverse monotonicity of $(DT_\varepsilon u_\varepsilon, R)$.

LEMMA 1 *([35, ch. 1])* *Let* $p,q \in C[0,1]$, $\varepsilon > 0$,

$$(3) \quad L_\varepsilon w = -\varepsilon w'' + pw' + qw, \quad w \in C^2[0,1].$$

The following conditions are equivalent:

(i) (L_ε, R) *is inverse monotone, i.e.*

$$L_\varepsilon w \geq 0, \ Rw \geq (0,0) \implies w \geq 0 \quad \text{for all } w \in C^2[0,1].$$

(ii) *The smallest eigenvalue of the eigenvalue problem*

$$L_\varepsilon w = \lambda w, \quad Rw = (0,0)$$

is positive.

(iii) *There exists* $e \in C^2[0,1]$ *with*

$$e \geq 0, \quad L_\varepsilon e \geq 0, \quad (L_\varepsilon, Re) \neq (0,0).$$

(iv) *Green's function* $G_\varepsilon(x,y)$ *corresponding to* (L_ε, R) *and the solutions* $g_\varepsilon, h_\varepsilon \in C^2[0,1]$ *of*

$$(4) \quad L_\varepsilon g_\varepsilon = L_\varepsilon h_\varepsilon = 0, \quad Rg_\varepsilon = (1,0), \quad Rh_\varepsilon = (0,1)$$

exist and are nonnegative.

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The following Theorem describes the sufficient conditions for the following properties of (P_ε), which can depend on ε .

PI: (P_ε) has a solution $u_\varepsilon \in C^2[0,1]$.

PII: (T_ε, R) is inverse monotone, i.e.

$$T_\varepsilon u \le T_\varepsilon v, \quad Ru \le Rv \Rightarrow u \le v \quad \text{for all } u,v \in C^2[0,1] .$$

Especially, (P_ε) has at most one solution.

PIII: The smallest eigenvalue of any eigenvalue problem

$$(5) \qquad (DT_\varepsilon u)w = \lambda w, \quad Rw = (0,0) ,$$

where $u \in C^2[0,1]$ is arbitrary, is positive. Especially, any solution of (P_ε) is stable. In some cases all eigenvalues of (5) have a common lower bound independent of u. This lower bound then is denoted by $\lambda_1(\varepsilon)$.

THEOREM 1 *Let the conditions of one of the following cases hold for all* $(x,u) \in [0,1] \times \mathbb{R}$ *where* $\alpha_o, \beta_o, \gamma_o \in \mathbb{R}$:

Case I: $\alpha(x,u) = \alpha(x) \ge \alpha_o > 0, \quad \beta_u(x,u) \ge \beta_o ,$
$\alpha_o^2 + 4\varepsilon\beta_o > 0, \quad \varepsilon > 0 .$

Case II: $\alpha(x,u) \ge \alpha_o > 0, \quad (\beta_u - \alpha_x)(x,u) \ge \gamma_o ,$
$\alpha_o^2 + 4\varepsilon\gamma_o > 0, \quad \varepsilon > 0 .$

Case III: $\alpha(x,u) = \alpha(x), \quad \beta_u(x,u) \ge 0, \quad \varepsilon > 0.$

Case IV: $(\beta_u - \alpha_x)(x,u) \ge 0, \quad \varepsilon > 0 .$

Then P II *and* P III *hold. In cases* I, I, III *also* P I *holds.*

In the following Theorems we give the stability inequalities for cases I-IV separately, partly under further conditions. In some cases we also give lower bounds for $\lambda_1(\varepsilon)$ where $\lambda_1(\varepsilon)$ is defined in P III. The norms

$$\|u\|_\infty = \max\{|u(x)| : 0 \le x \le 1\}, \quad \|u\|_1 = \int_0^1 |u(x)|dx$$

are used.

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THEOREM 2. *Assume case I with* $\beta_o \leq 0$. *For all* $u,v \in C^2[0,1]$ *holds*

$$|u(x) - v(x)| \leq e^{-2\beta_o/\alpha_o} \{H_\varepsilon ||T_\varepsilon u - T_\varepsilon v||_1 + |u(0) - v(0)| + |u(1) - v(1)| e^{(x-1)\alpha_o/\varepsilon}\}, \quad x \in [0,1]$$

with $H_\varepsilon = (\alpha_o^2 + 4\varepsilon\beta_o)^{-1/2}$.

Furthermore $\lambda_1(\varepsilon) \geq \alpha_o^2/4\varepsilon + \beta_o$.

THEOREM 3. *Assume case II with* $\gamma_o \leq 0$. *For all* $u,v \in C^2[0,1]$ *with* $u(0) = v(0)$ *holds*

$$|u(x) - v(x)| \leq e^{-2\gamma_o/\alpha_o} \{\bar{H}_\varepsilon ||T_\varepsilon u - T_\varepsilon v||_1 + |u(1) - v(1)| e^{(x-1)\alpha_o/\varepsilon}\}, \quad x \in [0,1] \tag{6}$$

with $\bar{H}_\varepsilon = (\alpha_o^2 + 4\varepsilon\gamma_o)^{-1/2}$.

Furthermore $\lambda_1(\varepsilon) \geq \alpha_o^2/4\varepsilon + \gamma_o$.

Assume Case IIa: the conditions of case II and $\alpha_1 \geq \alpha(x,u)$. Then for all $u,v \in C^2[0,1]$ holds

$$|u(x) - v(x)| \leq S_\varepsilon + e^{-2\gamma_o/\alpha_o} |u(0) - v(0)| (1 + \bar{H}_\varepsilon(\alpha_1 - \alpha_o))$$

where S_ε is the right hand side of (6).

REMARK 1.

a) The stability inequality of Theorem 2 was proved in |6| for all u,v with Ru = Rv.

b) Since $H_\varepsilon, \bar{H}_\varepsilon \to 1/\alpha_o < \infty$ for $\varepsilon \to 0^+$ we have stability uniformly in ε. Notice that $e^{(x-1)\alpha_o/\varepsilon}$ is a boundary layer function tending to zero for $x \in [0,1)$ and $\varepsilon \to 0^+$.

c) Since $\lambda_1(\varepsilon) \to \infty$ for $\varepsilon \to 0^+$ the solutions u_ε as stationary states of (1) become 'more stable' if ε gets smaller.

d) Similar results as stated in Theorems 2 and 3 hold if $\alpha(x,u) = \alpha(x) \leq -\alpha_o < 0$ or $\alpha(x,u) \leq -\alpha_o < 0$. Just use the transformations $x \to 1 - x$.

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Theorems 2 and 3 are only applicable to problems without turning points. The following Theorems will allow for turning point problems. First consider case III, and let

$$\alpha_{max} = \max\{\alpha(x) : x \in [0,1]\}, \quad \alpha_{min} = \min\{\alpha(x) : x \in [0,1]\} .$$

If $\alpha_{min} > 0$ or $\alpha_{max} < 0$ then Theorem 2 is applicable. Thus in the next Theorem assume $\alpha_{max} \geq 0 \geq \alpha_{min}$.

THEOREM 4. *Assume case III and let*

$$\alpha_o = \min\{\alpha_{max}, -\alpha_{min}\} \geq 0 .$$

For all $u,v \in C^2[0,1]$ *holds*

$$\|u-v\|_\infty \leq c_\varepsilon \|T_\varepsilon u - T_\varepsilon v\|_1 + \max\{|u(0)-v(0)|, |u(1)-v(1)|\}$$

with

(7) $c_\varepsilon = 1/\varepsilon$ *for* $\alpha_o = 0$, $c_\varepsilon = (e^{\alpha_o/\varepsilon} - 1)/\alpha_o$ *for* $\alpha_o > 0$.

Simple linear examples show that the stability constants c_ε can actually grow as $e^{\sigma/\varepsilon}$ $(\sigma > 0)$ for $\varepsilon \to 0^+$ under the general conditions of Theorem 4 (cf. |1|). The next two Theorems describe subcases of case III where better stability inequalities hold.

THEOREM 5. *Assume*

Case III a: $\alpha(x,u) = \alpha(x)$, $\beta_u(x,u) \geq \beta_o > 0$.

For all $u,v \in C^2[0,1]$ *holds*

$$\|u-v\|_\infty \leq \max\left\{\frac{1}{\beta_o} \|T_\varepsilon u - T_\varepsilon v\|_\infty, |u(0)-v(0)|, |u(1)-v(1)|\right\}.$$

Furthermore $\lambda_1(\varepsilon) \geq \beta_o > 0$.

THEOREM 6. *Assume*

Case III b: $\alpha(x,u) = \alpha(x)$, $\alpha(x) \geq 0$ *for* $0 \leq x \leq x_o$

$\alpha(x) \leq 0$ *for* $x_o \leq x \leq 1$, $\beta_u(x,u) \geq 0$.

For all $u,v \in C^2[0,1]$ *holds*

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$$||u-v||_\infty \leq \frac{1}{\varepsilon} \max\{x_o, 1-x_o\} \, ||T_\varepsilon u - T_\varepsilon v||_1$$
$$+ \max\{|u(0)-v(0)|, \, |u(1)-v(1)|\} .$$

Furthermore $\lambda_1(\varepsilon) \geq \frac{\pi^2}{4} \varepsilon \min\{x_o^{-2}, (1-x_o)^{-2}\}$.

Now consider subcases of case IV where $\alpha(x,u)$ can actually depend on u. Also, the conditions allow for $\alpha(x,u) = 0$ for some $(x,u) \in [0,1] \times \mathbb{R}$, thus turning points can occur.

THEOREM 7. *Assume*

Case IV a: $(\beta_u - \alpha_x)(x,u) \geq 0$ *and* $\alpha(x,u) \geq -\alpha_o$ *or* $\alpha(x,u) \leq \alpha_o$ *with* $\alpha_o \geq 0$.

For all $a,v \in C^2[0,1]$ *with* $Ru = Rv$ *holds*

$$||u-v||_\infty \leq c_\varepsilon ||T_\varepsilon u - T_\varepsilon v||_1$$

with c_ε *as in* (7). *The problem* (P_ε) *has a unique solution* u_ε *for all* $\varepsilon > 0$.

THEOREM 8. *Assume*

Case IV b:

$$(\beta_u - \alpha_x)(x,u) \geq \gamma_o > 0.$$

For all $u,v \in C^2[0,1]$ *with* $Ru = Rv$ *holds*

$$||u-v||_1 \leq \frac{1}{\gamma_o} ||T_\varepsilon u - T_\varepsilon v||_1 .$$

Furthermore $\lambda_1(\varepsilon) \geq \gamma_o > 0$.

THEOREM 9. *Assume*

Case IV c: $(\beta_u - \alpha_x)(x,u) \geq \gamma_o > 0$ *and* $-\alpha_o \leq \alpha(x,u) \leq \alpha_o$.

For all $u,v \in C^2[0,1]$ *with* $Ru = Rv$ *holds*

$$||u-v||_\infty \leq \frac{1}{4\varepsilon\gamma_o} (\alpha_o + (\alpha_o^2 + 4\varepsilon\gamma_o)^{1/2}) \, ||T_\varepsilon u - T_\varepsilon v||_1$$

Case IV c *is a subcase of* IV a *and* IV b, *thus* u_ε *exists and* $\lambda_1(\varepsilon) \geq \gamma_o > 0$.

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REMARK 2. a) Theorems 2-9 state stability inequalities for the nonlinear operator T_ε. The results can be stated in terms of stability for all linearized problems

$$(8) \qquad (DT_\varepsilon u)w = f, \quad Rw = (a,b), \quad f \in C[0,1], \ a,b \in \mathbb{R}$$

where $u \in C^2[0,1]$. E.g., let the conditions of Theorem 2 hold. Then for all $u \in C^2[0,1]$ the solutions w of the linear problem (8) exists and

$$|w(x)| \leq e^{-2\beta_o/\alpha_o}\{H_\varepsilon \|f\|_1 + |a| + |b| e^{(x-1)\alpha_o/\varepsilon}\}, \quad x \in [0,1].$$

b) For simplicity we have always assumed global conditions for the functions α and β. But it is apparent from the proofs below that all stability inequalities hold, if the assumptions on α and β are valid only between the chosen functions u and v occurring in the inequality.

2. THE PROOFS OF THEOREMS 1-9

Let the general conditions as stated in section 2 hold. For $u,v \in C^2[0,1]$ define the linear operator

$$\Delta T_\varepsilon(u,v) = \int_0^1 DT_\varepsilon(v+s(u-v))ds,$$

i.e. $\Delta T_\varepsilon(u,v)w = -\varepsilon w'' + pw' + qw$, $w \in C^2[0,1]$, where

$$(9) \qquad p(x) = \int_0^1 \alpha(x, z_s(x))ds, \quad x \in [0,1]$$

$$(10) \qquad q(x) = \int_0^1 \{\alpha_u(x, z_s(x))z_s'(x) + \beta_u(x, z_s(x))\}ds, \quad x \in [0,1]$$

with $z_s = v+s(u-v)$.

If $\alpha_x \in C[0,1] \times \mathbb{R})$ note that

$$\Delta T_\varepsilon(u,v)^* w = -\varepsilon w'' - pw' + rw, \quad w \in C^2[0,1],$$

where

$$(11) \qquad r(x) = (q - p')(x) = \int_0^1 (\beta_u - \alpha_x)(x, z_s(x))ds.$$

By the mean value theorem it holds that

$$T_\varepsilon u - T_\varepsilon v = \Delta T_\varepsilon(u,v)(u-v)$$

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and thus the following Lemma follows:

LEMMA 2. *Let* $L_\varepsilon = T_\varepsilon(u,v)$ *and assume*

$$L_\varepsilon w = 0, \quad Rw = (0,0) \Rightarrow w = 0. \tag{12}$$

Let $G_\varepsilon, g_\varepsilon, h_\varepsilon$ *be defined as in Lemma 1. Then*

$$(u-v)(x) = \int_0^1 G_\varepsilon(x,y)(T_\varepsilon u - T_\varepsilon v)(y)dy + (u-v)(0)g_\varepsilon(x) + (u-v)(1)h_\varepsilon(x), \quad x \in [0,1].$$

Now the proofs of the stability inequalities proceed as follows: The assumptions on α and β lead to estimates for the coefficients p, q, r of L_ε or L_ε^* uniformly for all $u,v \in C^2[0,1]$. Thus we study linear operators L_ε and we estimate G_ε, g_ε and h_ε only using inequalities for the coefficients of L_ε. Apart from L_ε other differential operators like $\hat{L}_\varepsilon$ etc. will occur. Then $\hat{G}_\varepsilon$ denotes Green's function corresponding to $(\hat{L}_\varepsilon, R)$ and $\hat{g}_\varepsilon$, $\hat{h}_\varepsilon$ are defined similarly as g_ε, h_ε (see 4)) with L_ε replaced by $\hat{L}_\varepsilon$. The following three Lemmas are elementary and only stated for convenience of later use.

LEMMA 3. *Let*

$$L_\varepsilon w = -\varepsilon w'' + pw' + qw, \quad w \in C^2[0,1] \tag{13}$$

with $p \in C^1[0,1]$, $q \in C[0,1]$ *and let*

$$p_\varepsilon(x) = \exp\left(\frac{1}{\varepsilon}\int_0^x p(s)ds\right). \tag{14}$$

Then

(i) $L_\varepsilon(p_\varepsilon w) = p_\varepsilon L_\varepsilon^* w$ *for all* $w \in C^2[0,1]$.

(ii) $G_\varepsilon(x,y) = p_\varepsilon(x)G_\varepsilon^*(x,y)/p_\varepsilon(y) = G_\varepsilon^*(y,x)$.

(iii) $g_\varepsilon = p_\varepsilon g_\varepsilon^*$, $h_\varepsilon = p_\varepsilon h_\varepsilon^*/p_\varepsilon(1)$.

In (ii) *and* (iii) *we assumed* (12).

LEMMA 4. *Let* L_ε *be given as in* (13) *with* $p,q \in C[0,1]$. *For* $\sigma \in \mathbb{R}$ *let*

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$$L_{\varepsilon,\sigma}w = -\varepsilon w'' + (p - 2\varepsilon\sigma)w' + (-\varepsilon\sigma^2 + p\sigma + q)w .$$

Then

(i) $L_\varepsilon(e^{\sigma x}w) = e^{\sigma x}L_{\varepsilon,\sigma}w, \quad w \in C^2[0,1].$

(ii) $G_\varepsilon(x,y) = e^{\sigma(x-y)}G_{\varepsilon,\sigma}(x,y).$

(iii) $g_\varepsilon = e^{\sigma x}g_{\varepsilon,\sigma}, h_\varepsilon = e^{\sigma(x-1)}h_{\varepsilon,\sigma} .$

In (ii) *and* (iii) *we assumed* (12).

LEMMA 5. *Let* L_ε *be given as in* (13) *with* $p,q \in C[0,1]$, *and let* $\hat{L}_\varepsilon w = -\varepsilon w'' + pw' + \hat{q}w$ *with* $\hat{q} \in C[0,1]$, $\hat{q} \leq q$. *If* $(\hat{L}_\varepsilon, R)$ *is inverse monotone, then* (L_ε, R) *is inverse monotone as well and*

$$0 \leq G_\varepsilon \leq \hat{G}_\varepsilon, \quad 0 \leq g_\varepsilon \leq \hat{g}_\varepsilon, \quad 0 \leq h_\varepsilon \leq \hat{h}_\varepsilon .$$

After these general preliminaries we first prove P II under the conditions of Theorem 1. Let $u,v \in C^2[0,1]$ and set

$$L_\varepsilon w = \Delta T_\varepsilon(u,v)w = -\varepsilon w'' + pw' + qw, \quad w \in C^2[0,1].$$

It holds

in case I: $p = \alpha \geq \alpha_o > 0$, $q \geq \beta_o$, $\alpha_o^2 + 4\varepsilon\beta_o > 0$,

in case II: $p \geq \alpha_o > 0$, $r = q - p' \geq \gamma_o$, $\alpha_o^2 + 4\varepsilon\gamma_o > 0$,

in case III: $q \geq 0$,

in case IV: $r = q - p' \geq 0$.

If we show that (L_ε, R) is inverse monotone, then P II follows. In case I set

$$\sigma = \frac{1}{2\varepsilon}\,(\alpha_o + (\alpha_o^2 + 4\varepsilon\beta_o)^{1/2}) > 0$$

and aplly Lemma 4:

$$L_\varepsilon(e^{\sigma x}) = e^{\sigma x}(-\varepsilon\sigma^2 + p\sigma + q) \geq e^{\sigma x}(-\varepsilon\sigma^2 + \alpha_o\sigma + \beta_o) = 0.$$

In case II set

$$\sigma = \frac{1}{2\varepsilon}\,(-\alpha_o - (\alpha_o^2 + 4\varepsilon\gamma_o)^{1/2}) < 0$$

and notice

$$L_\varepsilon^*(e^{\sigma x}) = e^{\sigma x}(-\varepsilon\sigma^2 - p\sigma + r) \geq e^{\sigma x}(-\varepsilon\sigma^2 - \alpha_o\sigma + \gamma_o) = 0.$$

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Let $\bar{e}(x) = 1$ for all $x \in [0,1]$. Then in case III holds $L_\varepsilon \bar{e} \geq 0$ and in case IV holds $L_\varepsilon^* \bar{e} \geq 0$. Thus in all cases (L_ε, R) is inverse monotone by Lemma 1, and P II follows.

To prove P III under the conditions of Theorem 1 we just notice that for any linearization (see (2))

$$(DT_\varepsilon u)w = -\varepsilon w'' + pw' + qw$$

the coefficients p,q and $r = q - p'$ satisfy the same inequalities as above. Thus with Lemma 1, (ii) the property P III follows.

P r o o f of Theorem 2. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$. Because of Lemma 2 we have to show that

$$G_\varepsilon(x,y) \leq e^{-2\beta_o/\alpha_o} H_\varepsilon,$$

$$g_\varepsilon(x) \leq e^{-2\beta_o/\alpha_o}, h_\varepsilon(x) \leq e^{-2\beta_o/\alpha_o + (x-1)\alpha_o/\varepsilon}. \tag{15}$$

a) First assume $\beta_o = 0$, i.e. $q \geq 0$. If $\hat{L}_\varepsilon w = -\varepsilon w'' + pw'$, then elementary calculation shows (with p_ε as in (14)):

$$\hat{G}_\varepsilon(x,y) = (\varepsilon p_\varepsilon(y) \int_o^1 p_\varepsilon ds)^{-1} \cdot \begin{cases} \int_o^x p_\varepsilon ds \int_y^1 p_\varepsilon ds, & x \leq y, \\ \int_o^y p_\varepsilon ds \int_x^1 p_\varepsilon ds, & y \leq x, \end{cases} \tag{16}$$

Since $p \geq \alpha_o > 0$ we have

$$\hat{G}_\varepsilon(x,y) \leq \hat{G}_\varepsilon(y,y) \leq \frac{1}{\varepsilon p_\varepsilon(y)} \int_o^y p_\varepsilon ds$$

$$= \frac{1}{\varepsilon} \int_o^y \exp\left(-\frac{1}{\varepsilon} \int_s^y p d\sigma\right) ds \leq \frac{1}{\varepsilon} \int_o^y e^{-(y-s)\alpha_o/\varepsilon} ds$$

$$\leq (1 - e^{-\alpha_o/\varepsilon})/\alpha_o < 1/\alpha_o .$$

Because of Lemma 5 we thus have $G_\varepsilon(x,y) \leq 1/\alpha_o$.

b) Let $\beta_o < 0$, $\sigma = \frac{1}{2\varepsilon}(\alpha_o - (\alpha_o^2 + 4\varepsilon\beta_o)^{1/2}) > 0$.

Use the transformation described in Lemma 4 and notice that the proved part a) is applicable to $L_{\varepsilon,\sigma}$ with

$$\alpha_{o,\sigma} = \alpha_o - 2\varepsilon\sigma = (\alpha_o^2 + 4\varepsilon\beta_o)^{1/2} .$$

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Lemma 4 yields

$$G_\varepsilon(x,y) \leq e^{\sigma}(\alpha_o^2 + 4\varepsilon\beta_o)^{-1/2} \leq e^{-2\beta_o/\alpha_o} H_\varepsilon$$

where the last estimate follows from

$$\sigma = \frac{1}{2\varepsilon}\,(\alpha_o - \alpha_o(1 + 4\varepsilon\beta_o/\alpha_o^2)^{1/2})$$

$$\leq \frac{1}{2\varepsilon}\,(\alpha_o - \alpha_o(1 + 4\varepsilon\beta_o/\alpha_o^2)) = -2\beta_o/\alpha_o .$$

c) Since $\beta_o \leq 0$, both roots

$$\sigma_{1,2} = \frac{1}{2\varepsilon}\,(\alpha_o \pm (\alpha_o^2 + 4\varepsilon\beta_o)^{1/2})$$

of $-\varepsilon\sigma^2 + \alpha_o\sigma + \beta_o = 0$ are nonnegative, and thus $p \geq \alpha_o$, $q \geq \beta_o$ implies $L_\varepsilon(e^{\sigma_i x}) \geq 0$. Since $R(e^{\sigma_2 x}) \geq (1,0)$, $R(e^{\sigma_1(x-1)} \geq (0,1)$ it follows that $g_\varepsilon(x) \leq e^{\sigma_2 x}$, $h_\varepsilon(x) \leq e^{\sigma_1(x-1)}$, which leads to to (15) since $\sigma_2 \leq -2\beta_o/\alpha_o$, $\sigma_1 = \alpha_o/\varepsilon - \sigma_2 \geq \alpha_o/\varepsilon + 2\beta_o/\alpha_o$.

d) To show $\lambda_1(\varepsilon) \geq \alpha_o^2/4\varepsilon + \beta_o$, just consider $e(x) = \exp(\alpha_o x/2\varepsilon)$ and notice

$$(DT_\varepsilon u)e \geq (\alpha_o^2/4\varepsilon + \beta_o)e .$$

Thus $(DT_\varepsilon u - (\alpha_o^2/4\varepsilon + \beta_o), R)$ in inverse monotone, and by Lemma 1 the assertion follows.

P r o o f of Theorem 3. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$, $L_\varepsilon^* w = -\varepsilon w'' - pw' + rw$ with $r \geq \gamma_o$. We have to show that

(17) $$G_\varepsilon(x,y) \leq e^{-2\gamma_o/\alpha_o} \bar{H}_\varepsilon ,$$

(18) $$h_\varepsilon(x) \leq e^{-2\gamma_o/\alpha_o + (x-1)\alpha_o/\varepsilon} ,$$

and in case II a

(19) $$g_\varepsilon(x) \leq e^{-2\gamma_o/\alpha_o}(1 + \bar{H}_\varepsilon(\alpha_1 - \alpha_o)) .$$

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a) By the proof of Theorem 2 it follows that $G_\varepsilon^*(x,y) \le e^{-2\gamma_o/\alpha_o}\, \bar{H}_\varepsilon$, which proves (17).

b) To show (18) and (19) first assume $\gamma_o = 0$. With p_ε defined by (14) consider

$$\bar{g}_\varepsilon(x) = p_\varepsilon(x) \int_x^1 \frac{ds}{p_\varepsilon(s)} \Big/ \int_0^1 \frac{ds}{p_\varepsilon(s)} ,$$

$$\bar{h}_\varepsilon(x) = \frac{p_\varepsilon(x)}{p_\varepsilon(1)} \int_0^x \frac{ds}{p_\varepsilon(s)} \Big/ \int_0^1 \frac{ds}{p_\varepsilon(s)} .$$

With Lemma 3, (i) it follows that $L_\varepsilon \bar{g}_\varepsilon \ge 0$ and $L_\varepsilon \bar{h}_\varepsilon \ge 0$. Since $R\bar{g}_\varepsilon = (1,0)$, $R\bar{h}_\varepsilon = (0,1)$ we thus have $g_\varepsilon \le \bar{g}_\varepsilon$, $h_\varepsilon \le \bar{h}_\varepsilon$. Now $p \ge \alpha_o > 0$ implies

$$\bar{h}_\varepsilon(x) \le p_\varepsilon(x) / p_\varepsilon(1) = \exp\Big(-\frac{1}{\varepsilon}\int_x^1 p(s)ds\Big) \le e^{(x-1)\alpha_o/\varepsilon} .$$

If $0 \le \alpha_o \le p \le \alpha_1$, then

$$p_\varepsilon(x) \int_x^1 \frac{ds}{p_\varepsilon(s)} = \int_x^1 \exp\Big(-\frac{1}{\varepsilon}\int_x^s p(\sigma)d\sigma\Big)ds \le \varepsilon(1 - e^{-\alpha_o/\varepsilon}) / \alpha_o ,$$

and

$$1 / p_\varepsilon(s) = \exp\Big(-\frac{1}{\varepsilon}\int_0^s p(\sigma)d\sigma\Big) \ge e^{-s\alpha_1/\varepsilon} ,$$

$$\int_0^1 \frac{ds}{p_\varepsilon(s)} \ge \varepsilon(1 - e^{-\alpha_1/\varepsilon}) / \alpha_1 .$$

If follows that $\bar{g}_\varepsilon \le \alpha_1 / \alpha_o$, and (18), (19) are proved for $\gamma_o = 0$.

c) Let $\gamma_o < 0$, $\sigma = \frac{1}{2\varepsilon}(\alpha_o - (\alpha_o^2 + 4\varepsilon\gamma_o)^{1/2}) > 0$.

The proved part b) is applicable to $L_{\varepsilon,\sigma}$ as defined in Lemma 4, and we get

$$g_{\varepsilon,\sigma}(x) \le (\alpha_1 - 2\varepsilon\sigma) / (\alpha_o - 2\varepsilon\sigma) = 1 + \bar{H}_\varepsilon(\alpha_1 - \alpha_o) ,$$

$$h_{\varepsilon,\sigma}(x) \le e^{(x-1)(\alpha_o - 2\varepsilon\sigma)/\varepsilon} ,$$

$$g_\varepsilon(x) \le e^\sigma(1 + \bar{H}_\varepsilon(\alpha_1 - \alpha_o)) ,$$

$$h_\varepsilon(x) \le e^{\sigma + (x-1)\alpha_o/\varepsilon} ,$$

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and (18), (19) follow $\sigma \leq -2\gamma_o / \alpha_o$.

d) The assertion $\lambda_1(\varepsilon) \geq \alpha_o^2 / 4\varepsilon + \gamma_o$ follows as in the proof of of Theorem 2 since DT_ε and $(DT_\varepsilon u)^*$ have the same eigenvalues.

P r o o f of Theorem 4. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$, $\hat{L}_\varepsilon w = -\varepsilon w'' + pw'$. By Lemma 5 holds $G_\varepsilon(x,y) \leq \hat{G}_\varepsilon(x,y)$, and as in the proof Theorem 2 it follows that $\hat{G}_\varepsilon(x,y) \leq c_\varepsilon$. Since $L_\varepsilon \bar{e} \geq 0$, $R\bar{e} = (1,1)$ we have $g_\varepsilon + h_\varepsilon \leq \bar{e}$. With Lemma 1 the Theorem is proved.

P r o o f of Theorem 5. Let $L_\varepsilon w = \Delta T_\varepsilon(u,v)w = -\varepsilon w'' + pw' + qw$ with $p = \alpha$, $q \geq \beta_o > 0$. Set

$$c = \max \{ \frac{1}{\beta_o} \| T_\varepsilon u - T_\varepsilon v \|_\infty , |u(0) - v(0)|, |u(1) - v(1)| \} ,$$

then $L_\varepsilon(c\bar{e} \pm (u-v)) \geq c\beta_o \bar{e} \pm (T_\varepsilon u - T_\varepsilon v) \geq 0$,

$$R(c\bar{e} \pm (u-v)) \geq (0,0) ,$$

thus $\pm(u-v) \leq c\bar{e}$. The inequality $\lambda_1(\varepsilon) \geq \beta_o$ is obvious since $((DT_\varepsilon u) - \beta_o, R)$ is inverse-monotone.

P r o o f of Theorem 6.

a) We have to show that $\hat{G}_\varepsilon(x,y) \leq \frac{1}{\varepsilon} \max\{x_o, 1 - x_o\}$ where $\hat{G}_\varepsilon$ is given in (16) and

$$\hat{G}_\varepsilon(x,y) \leq \hat{G}_\varepsilon(y,y) = \int_o^y p_\varepsilon ds \int_y^1 p_\varepsilon ds \, / (\varepsilon p_\varepsilon(y) \int_o^1 p_\varepsilon ds) .$$

Since $p_\varepsilon(s)$ is monotonically increasing for $0 \leq s \leq x_o$ and decreasing for $x_o \leq s \leq 1$ we have for $0 \leq y \leq x_o$

$$\hat{G}_\varepsilon(y,y) \leq \frac{1}{\varepsilon} \int_o^y p_\varepsilon(s) ds \, / p_\varepsilon(y) \leq y / \varepsilon \leq x_o / \varepsilon ,$$

and similarly for $x_o \leq y \leq 1$

$$\hat{G}_\varepsilon(y,y) \leq \frac{1}{\varepsilon} \int_y^1 p_\varepsilon(s) ds \, / p_\varepsilon(y) \leq (1-y)/\varepsilon \leq (1-x_o) / \varepsilon .$$

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b) To prove the statement about $\lambda_1(\varepsilon)$ first assume $\frac{1}{2} \le x_o \le 1$ and let

$$e(x) = \sin\left(\frac{\pi x}{2x_o}\right). \tag{20}$$

Since $e \ge 0$, $e'(x) \ge 0$ in $[0,x_o]$, $e'(x) \le 0$ in $[x_o,1]$ it holds that

$$(DT_\varepsilon u)e \ge -\varepsilon e'' = \frac{\pi^2}{4}\varepsilon x_o^{-2} e = \frac{\pi^2}{4}\varepsilon \min\{x_o^{-2}, (1-x_o)^{-2}\} e .$$

In case that $0 \le x \le x_o$, consider instead of (20) the function $e(x) = \sin\left(\frac{\pi(1-x)}{2(1-x_o)}\right)$.

P r o o f of Theorem 7. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$, $L_\varepsilon^* w = -\varepsilon w'' - pw' + rw$ with $r \ge 0$. As in the proof of Theorem 2 it follows that $G_\varepsilon^*(x,y) \le c_\varepsilon$, which proves the stability inequality. The existence result follows from Theorem 10 below.

P r o o f of Theorem 8. Let L_ε^* be defined as above. Since $r \ge \gamma_o > 0$ we have with $\bar{e}(x) \equiv 1$:

$$L_\varepsilon^* \bar{e} \ge \gamma_o \bar{e}, \text{ thus } G_\varepsilon^* \bar{e} \le \frac{1}{\gamma_o}\bar{e}, \text{ i.e. } \int_0^1 G_\varepsilon(x,y)dx \le \frac{1}{\gamma_o},$$

and the stability inequality follows. The smallest eigenvalue of the eigenvalue problem

$$(DT_\varepsilon u)^* w = \lambda w, \quad Rw = (0,0)$$

is $> \gamma_o$, thus $\lambda_1(\varepsilon) \ge \gamma_o$.

P r o o f of Theorem 9. Let $L_\varepsilon = \Delta T_\varepsilon(u,v)$, $L_\varepsilon^* w = -\varepsilon w'' - pw' + rw$. It holds that $r \ge \gamma_o > 0$ and $-\alpha_o \le p \le \alpha_o$, and we have to show the estimate

$$G_\varepsilon^*(x,y) \le \frac{1}{4\varepsilon\gamma_o}\left(\alpha_o + (\alpha_o^2 + 4\varepsilon\gamma_o)^{1/2}\right) \tag{21}$$

Now G_ε^* has a representation

$$G_\varepsilon^*(x,y) = \frac{1}{\varepsilon W_\varepsilon(y)} \cdot \begin{cases} \phi_\varepsilon(x)\psi_\varepsilon(y), & x \le y \\ \phi_\varepsilon(y)\psi_\varepsilon(x), & y \le x \end{cases}$$

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where $L_\varepsilon^* \phi_\varepsilon = L_\varepsilon^* \psi_\varepsilon = 0$, $\phi_\varepsilon(0) = \psi_\varepsilon(1) = 0$, $\phi_\varepsilon'(0) = -\psi_\varepsilon(1) = 1$, $W_\varepsilon = \phi_\varepsilon' \psi_\varepsilon - \psi_\varepsilon' \phi_\varepsilon$. Since $\phi_\varepsilon' > 0 > \psi_\varepsilon'$, $W_\varepsilon > 0$ we have

$$G_\varepsilon^*(x,y) \leq G_\varepsilon^*(y,y) = \frac{1}{\varepsilon} \frac{\phi_\varepsilon \psi_\varepsilon}{\phi_\varepsilon' \psi_\varepsilon - \psi_\varepsilon' \phi_\varepsilon}(y) ,$$

and thus for $0 < y < 1$:

$$(G_\varepsilon^*(y,y))^{-1} = \varepsilon \left(\frac{\phi_\varepsilon'}{\phi_\varepsilon} - \frac{\psi_\varepsilon'}{\psi_\varepsilon} \right)(y) .$$

The function $p_\varepsilon = \phi_\varepsilon / \phi_\varepsilon' \in C^1[0,1]$ satisfies the Riccati equation

$$\varepsilon \rho' - \varepsilon \rho - p\rho + r\rho^2 = 0, \quad \rho(0) = 0. \tag{22}$$

Since the constant $e(x) = c$ is an upper solution of (22) for

$$c = \frac{1}{2\gamma_0} \left(\alpha_0 + (\alpha_0^2 + 4\varepsilon\gamma_0)^{1/2} \right) > 0$$

we conclude |33, ch. 2| that $\rho_\varepsilon \leq c$. Similarly, $-\psi_\varepsilon / \psi_\varepsilon' \leq c$, thus

$$(G_\varepsilon^*(y,y))^{-1} \geq 2\varepsilon / c$$

and (21) follows.

P r o o f of the existence statements

In many cases the existence of a solution of (P_ε) is most easily established by the Nagumo-Lemma |18,26| which implies

LEMMA 6. *Let* $\alpha, \beta \in C([0,1] \times \mathbb{R})$, $\varepsilon > 0$, *and assume for two functions* $u, v \in C^2[0,1]$ *holds*

$$u \leq v, \quad T_\varepsilon u \geq 0 \geq T_\varepsilon v, \quad Ru \leq (A,B) \leq Rv .$$

Then (P_ε) *has a solution* $u_\varepsilon \in C^2[0,1]$ *with* $u \leq u_\varepsilon \leq v$.

Also notice that in cases I and II, where α is independent of u, the solvability of (P_ε) follows from |3, Th. 7.6|. (The value of $\alpha(L_2, K_2)$ in |3, Th. 7.6)| is ∞ under our conditions). The existence statements for all our cases I, II, III, IVa follow from the next Theorem where we just require a stability

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inequality in the maximum norm.

THEOREM 10*: *Let* $\alpha,\beta \in C([0,1] \times \mathbb{R})$,

$$Tu = -u'' + \alpha(x,u)u' + \beta(x,u),\ u \in U = \{v \in C^2[0,1],\ Rv = 0\}.$$

Suppose for some $k > 0$

$$\|u-v\|_\infty \leq k\|Tu - Tv\|_\infty \quad \forall\ u,v \in U. \tag{23}$$

Then T *is a homemorphism from* $(U, \|\ \|_{2,\infty})$ *onto* $(C[0,1], \|\ \|_\infty)$ *with* $\|u\|_{2,\infty} = \|u\|_\infty + \|u'\|_\infty + \|u''\|_\infty$.

P r o o f. a. Let $Fu = \alpha(\cdot,u)u' + \beta(\cdot,u)$ for $u \in U$, and let G denote Green's function to $(-u'',R)$. Thus for all $r \in C[0,1]$ holds $Tu = r$, $u \in U \Leftrightarrow u + GFu = Gr$, $u \in U$. Since F is completely continuous from U into $C[0,1]$ and since G is bounded from $C[0,1]$ into U the operator

$$K = GF : U \to U$$

is completely continuous.

Also, $I + K$ is injective on U, since

$$u + GFu = v + GFv,\ u,v \in U \Rightarrow Tu = Tv \Rightarrow u = v \quad \text{by (23)}.$$

Now the Brouwer-Schauder Theorem of the invariance of domain (cf. |34, Satz 15.3|) yields: $(I+K)(U)$ is open in U and

$$(I+K)^{-1} : (I+K)(U) \subset U \to U$$

is continuous.

b, It remains to show that $(I+K)(U)$ is closed in U. To prove this let

$$u_n \in U,\quad v_n = u_n + Ku_n \to \bar{v} \in U \quad \text{for} \quad n \to \infty.$$

Then $v_n - v_m = u_n + Ku_n - u_m - Ku_m$ implies $-v_n'' + v_m'' = Tu_n - Tu_m$, thus

$$\|Tu_n - Tu_m\|_\infty \to 0 \quad \text{for} \quad n,m \to \infty.$$

*) I would like to thank Dr. W.-J. Beyn for helpful remarks and discussions concerning this result.

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(23) yields $u_n \to \bar{u} \in C[0,1]$ for $n \to \infty$ with respect to $\|\ \|_\infty$. Define the linear operator

$$Lw = -w'' + \alpha(\cdot,\bar{u}(\cdot))w' \ , \quad w \in U \ ,$$

and solve $L\bar{w} = -\bar{v}''(\cdot) - \beta(\cdot,\bar{u}(\cdot))$, $\bar{w} \in U$. The linear operator L satisfies a stability inequality on U:

$$\|w\|_{2,\infty} \leq \bar{k}\|Lw\|_\infty \quad \text{for all} \quad w \in U \ ,$$

and taking $w = u_n - \bar{w}$ we find

$$\begin{aligned}
\|u_n - \bar{w}\|_{2,\infty} &\leq \bar{k}\|Lu_n - L\bar{w}\|_\infty \\
&= \bar{k}\|-u_n'' + \alpha(\cdot,\bar{u})u_n' - L\bar{w}\|_\infty \\
&= \bar{k}\|-v_n'' - \alpha(\cdot,u_n)u_n' - \beta(\cdot,u_n) + \alpha(\cdot,\bar{u})u_n' \\
&\quad + \bar{v}'' + \beta(\cdot,\bar{u})\|_\infty \leq \eta_n + \|(\alpha(\cdot,\bar{u}) - \alpha(\cdot,u_n))(u_n' - \bar{w}')\|_\infty \\
&\quad + \|(\alpha(\cdot,\bar{u}) - \alpha(\cdot,u_n))\bar{w}'\|_\infty \leq \bar{\eta}_n\|u_n' - \bar{w}'\|_\infty + \bar{\bar{\eta}}_n
\end{aligned}$$

where η_n, $\bar{\eta}_n$, $\bar{\bar{\eta}}_n \to 0$ for $n \to \infty$. Thus $u_n \to \bar{w}$ in U, and thus

$$\bar{v} = \bar{w} + K\bar{w} \in (I+K)(U).$$

c. $I + K = I + GF$ is a homeomorphism from U onto itself, and therefore

$$T = -D^2(I + GF) : U \to C[0,1]$$

is also a homeomorphism.

4. APPLICATIONS AND FURTHER REMARKS

In this section we consider a class of examples and characterize the Minty-monotonicity of T_ε.

EXAMPLE.

(24) $$T_\varepsilon u \equiv -\varepsilon u'' + uu' + \mu u = \gamma(x), u(0) = A, \quad u(1) = B$$

with $\varepsilon > 0$, μ, $A, B \in \mathbb{R}$, $\gamma \in C[0,1]$. These examples include the stationary Burgers equation $-\varepsilon u'' + uu' = 0$ (cf. |4,7,28|) and Cochran's equation $-\varepsilon u'' + uu' + u = 0$ (cf. |8,12,27|). With Lemma 6 it follows that (24) has a solution u_ε. (Upper and lower solutions-

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even independent of ε - of the form

$$v(x) = cx + d, \quad c > -\mu, \quad d \in \mathbb{R}$$

are easily computed.) If $\mu \geq 0$, then Theorem 1, case IV shows the uniqueness and monotone dependence of u_ε upon γ, A and B. For $\mu < 0$ a solution of (24) will not be unique in general. Take e.g. $\gamma = 0$, $A = B = 0$. Then $u_\varepsilon = 0$ is a solution for all ε, and bifurcation occurs at $\varepsilon = -\mu/(n^2\pi^2) > 0$, $n \in \mathbb{N}$. Assume $\mu \geq 0$ and $\max\{\|\gamma\|_\infty, |A|, |B|\} \leq M$. Then $\|u_\varepsilon\|_\infty \leq M + 1$ since $u(x) = x - M - 1$ is a lower solution and $v(x) = x + M$ is an upper solution for (24). Thus it is sufficient to obtain stability inequalities for all $u, v \in C^2[0,1]$ with $\|u\|_\infty \leq \alpha_o$, $\|v\|_\infty \leq \alpha_o$. Because of Remark 2b, Theorem 7 and Theorem 9 the following holds:

Let $u, v \in C^2[0,1]$, $\|u\|_\infty \leq \alpha_o$, $\|v\|_\infty \leq \alpha_o$, $Ru = Rv$, and let T_ε be given as in (24). Then

$$\|u - v\|_\infty \leq s_{\varepsilon\mu} \|T_\varepsilon u - T_\varepsilon v\|_1$$

where

$$s_{\varepsilon\mu} = \begin{cases} \frac{1}{4\varepsilon\mu}\left(\alpha_o + (\alpha_o^2 + 4\varepsilon\mu)^{1/2}\right) & \text{for } \mu > 0, \\ \frac{1}{\alpha_o}\left(e^{\alpha_o/\varepsilon} - 1\right) & \text{for } \mu = 0. \end{cases}$$

Actually, for $\mu = 0$ the growth of the stability constants as $e^{\sigma/\varepsilon}$ $(\sigma > 0)$ for $\varepsilon \to 0^+$ cannot be improved. To see this, consider Burgers equation

$$(25) \qquad -\varepsilon u'' + uu' = 0, \quad u(0) = 1, \quad u(1) = -1 + \eta \quad .$$

(25) can be solved analytically (cf. [28, pp. 9-12]), and it follows that the solution for $\eta = e^{-\sigma/\varepsilon}$ with suitable $\sigma > 0$ differs from the solution for $\eta = 0$ by at least one. A transformation of (25) to homogeneous boundary conditions then proves our statement.

The following Theorem concerns the monotonicity of T_ε in the sense of Minty [25]. Here T is considered as an operator mapping $U = \overset{o}{W}{}^2[0,1]$ into $L_2[0,1]$, and

$$(u,v) = \int_0^1 uv\,dx \quad \text{for} \quad u, v \in L_2[0,1] \quad .$$

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THEOREM 11. *Assume*

$$\alpha, \alpha_x, \alpha_u, \beta, \beta_u \in C([0,1] \times \mathbb{R}) \quad \text{and} \quad \varepsilon > 0,$$

(i) *If*

$$\alpha(x,u) = \alpha(x), \beta_u(x,u) - \frac{1}{2}\alpha_x(x) \geq 0 \tag{26}$$

then

$$(T_\varepsilon u - T_\varepsilon v, u-v) \geq \varepsilon\pi^2 (u-v, u-v) \quad \forall u, v \in U,$$

i.e. T_ε *is uniformly Minty-monotone for each* $\varepsilon > 0$.

(ii) *If (26) is violated then* T_ε *is not Minty-monotone for* $0 < \varepsilon \leq \varepsilon_o$.

P r o o f. a) Let $L_\varepsilon w = -\varepsilon w'' + pw' + qw$, $w \in U$ with $p \in C^1[0,1]$, $q \in C[0,1]$. Since

$$(pw', w) = (w, -p'w - pw'), \text{ i.e. } (pw', w) = -\frac{1}{2}(p'w, w),$$

we have

$$(L_\varepsilon w, w) = \varepsilon(w', w') + ((q - \frac{1}{2}p')w, w) . \tag{27}$$

If $q - \frac{1}{2}p' \geq 0$ then

$$(L_\varepsilon w, w) \geq \varepsilon\pi^2 (w, w) \quad \text{for all} \quad w \in U,$$

and taking $L_\varepsilon = \Delta T_\varepsilon(u,v)$ we have proved (i).

b) First note that T_ε is Minty-monotone iff

$$((DT_\varepsilon u)w, w) \geq 0 \quad \text{for all} \quad u, w \in U.$$

Now $(DT_\varepsilon u)w = -\varepsilon w'' + pw' + qw$ with

$$p(x) = \alpha(x, u(x)), \quad q(x) = \beta_u(x, u(x)) + \alpha_u(x, u(x))u'(x), \tag{28}$$

and if $(q - \frac{1}{2}p')(x) < 0$ for some $x \in [0,1]$, then $L_\varepsilon = DT_\varepsilon u$ is not Minty-monotone for $0 < \varepsilon \leq \varepsilon_o$ because of (27). Here it holds that

$$(q - \frac{1}{2}p')(x) = \beta_u(x, u(x)) - \frac{1}{2}\alpha_x(x, u(x)) + \frac{1}{2}\alpha_u(x, u(x))u'(x).$$

If $\alpha_u(x_o, u_o) \neq 0$ for some $(x_o, u_o) \in (0,1) \times \mathbb{R}$, then solve the initial value problem

$$\beta_u(x, \bar{u}) - \frac{1}{2}\alpha_x(x, \bar{u}) + \frac{1}{2}\alpha_u(x, \bar{u})\bar{u}' = -1, \quad \bar{u}(x_o) = u_o$$

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for $x \in (x_o - \sigma, x_o + \sigma)$. A simple process of approximation and extension thus leads to a function $u \in U$, for which the functions p and q defined in (28) fulfill

$$(q - \frac{1}{2} p')(x_o) < 0.$$

The case $\alpha_u \equiv 0$, $\beta_u(x_o,u_o) - \frac{1}{2} \alpha_x(x_o) < 0$ for some (x_o,u_o) is treated similarly.

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REZIME

OSOBINE STABILNOSTI I MONOTONIJE ZA ČVRST KVAZILINEARAN GRANIČNI PROBLEM

U ovom radu ispituju se osobine stabilnosti i monotonije jedne klase kvazilinearnog graničnog problema.

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ON A NUMERICAL SOLUTION OF THE BOUNDARY VALUE PROBLEM USING AN OPTIMAL NUMERICAL DIFFERENTIATION

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ABSTRACT

In this paper we consider a numerical solution of two boundary value problems: (BVP1) $-u''(x) = f(x,u)$, $u(0) = \gamma_0$, $u(1) = \gamma_1$, and (BVP2) $-u''(x) - q(x)u(x) = f(x)$, $u(0) = \gamma_0$, $u(1) = \gamma_1$, using the authors' four-point rule of degree 3 for the second derivative. The discretisation meshes depend upon this rule. The discrete problem to (BVP1) has the usual form, but the discrete problem to (BVP2) has an unusual form. Under certain assumptions, our schemes have a third order of convergence.

1. INTRODUCTION

We consider two boundary value problems (BVP1) and (BVP2).

(BVP1) $\quad -u''(x) = f(x,u)$ on $[0,1]$, $\quad u(0) = \gamma_0$, $\quad u(1) = \gamma_1$,

where $f \in C^3([0,1] \times \mathbb{R})$ and $\gamma_0, \gamma_1 \in \mathbb{R}$. The nonlinearity $f(t,v)$ is assumed to satisfy the Lipschitz condition

(L) $\quad q(v-w) \leq f(x,v) - f(x,w) \leq \mu(v-w)$, $v,w \in \mathbb{R}$, $v \geq w$, $x \in [0,1]$

for some reals $q, \mu \in \mathbb{R}$, where

(1) $\quad -h^{-2} q_0 \leq \mu < \lambda_0$, $\quad q_0 = 3(\sqrt{5} - 1)$, $0 < \lambda_0 < 8$,

and we let $h = (1 + n\frac{3+\sqrt{5}}{2})^{-1}$ be so small, i.e. $n \in \mathbb{N}$ so great, that

(2) $\quad -h^{-2} q_0 < \frac{1}{2}(\lambda_0 + q)$

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(BVP2) $-u''-q(x)u = f(x)$ on $[0,1]$, $u(0) = \beta_0$, $u(1) = \beta$,

where $q,f \in C^3[0,1]$, $\beta_0, \beta_1 \in \mathbb{R}$. We aasume

$$0 \leq q(x) \leq \mu < 8 \quad \text{on } [0,1]. \tag{3}$$

The assumptions for (BVP1) are similar to the assumptions from |1|, but here q_0 and λ_0 are different. The parametar h depends on the rule we used for the discretisation of (BVP1). For q(x) from (BVP2) a usual assumption is $q(x) \leq 0$ in $[0,1]$. But, our scheme for (BVP2) has a special form and we have the assumption $0 \leq q(x) < 8$.

The proofs in section 3 are based upon the results of the type in |1|,|2|,|7|,|8|. (5)

2. FINITE DIFFERENCE SCHEMES

In this section we shall describe the schemes which we are going to discuss. It is convenient to distinguish (BVP1) and (BVP2). First we shall consider (BVP1).

For $u \in C^5[0,1]$, the authors' four-point rule of degree for the second derivative, see |4|, is

$$-u''(x) = h^{-2}\left[af(x\pm h) + bf(x) + cf\left(x \mp \frac{1+\sqrt{5}}{2}h\right) + df\left(x \mp \frac{3+\sqrt{5}}{2}h\right)\right] + O(h^3), \tag{4}$$

where

$$a = -\frac{2\sqrt{5}}{5}, \quad b = 6 - 2\sqrt{5}, \quad c = -(3-\sqrt{5}), \quad d = \frac{7\sqrt{5}}{5} - 3.$$

We let for $n \in \mathbb{N}$,

$$I_h = \{x_{2i} = i\,\frac{3+\sqrt{5}}{2}\,h,\ x_{2i+1} = h + x_{2i},\ i=0,1,\ldots,n\},$$

where $h = \left(1 + n\,\frac{3+\sqrt{5}}{2}\right)^{-1}$. Now we can form the descrete analogue to (BVP1) by using (4):

$$(L_h y)_i = \begin{cases} \gamma_0, & i = 0, \\ 0, & i = 1,2,\ldots,2n, \\ \gamma_1, & i = 2n+1, \end{cases} \tag{DBVP 1}$$

where

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$$(L_h y)_i = \begin{cases} y_i , & i=0,2n+1, \\ h^{-2}[dy_{i-2}+cy_{i-1}+by_i+ay_{i+1}]-f(x_i,y_i), & i=2m,\ m=1,\dots,n, \\ h^{-2}[ay_{i-1}+by_i+cy_{i+1}+dy_{i+2}]-f(x_i,y_i), & i=2m+1,\ m=0,1,\dots n-1. \end{cases}$$

We can write the discrete problem to (BVP1) in the canonical form

$$A_h y - F_h y = 0,$$

where

$$A_h = h^{-2}\begin{bmatrix} h^2 & & & & & & & & \\ a & b & c & d & & & & & \\ d & c & b & a & & & & & \\ & & a & b & c & d & & & \\ & & d & c & b & a & & & \\ & & & \ddots & \ddots & \ddots & \ddots & & \\ & & & & d & c & b & a & \\ & & & & & & a & b & c & d \\ & & & & & & d & c & b & a \\ & & & & & & & & & h^2 \end{bmatrix} \in \mathbb{R}^{2n+1,2n+2} \tag{5}$$

and F_h is a nonlinear mapping of R^{2n+2} into itself which assigns to $y = [y_0, y_1, \dots, y_{2n+1}]^T$ the element $F_h y = [\gamma_0, f_1, f_2, \dots, f_{2n}, \gamma_1]^T$, with $f_i = f(x_i, y_i)$.

For the discretisation of (BVP2) we use our four-point rule of degree 3 for the second derivative, [5],

$$-u''(x) = k^{-2}\left[Af\left(x \mp \frac{9-\sqrt{15}}{6}k\right) + Bf\left(x \pm \frac{\sqrt{15}-3}{6}k\right) + Cf\left(x \pm \frac{3+\sqrt{15}}{6}k\right) + Df\left(x \pm \frac{9+\sqrt{15}}{6}k\right)\right] + O(k^3), \quad u \in C^5[0,1], \tag{6}$$

where

$$A = -\frac{3+\sqrt{15}}{6}, \quad B = \frac{1+\sqrt{15}}{2}, \quad C = -\frac{\sqrt{15}-1}{2}, \quad D = \frac{\sqrt{15}-3}{6}.$$

In this case, the mesh is

$$J_k = \{x_{2i} = ik,\ x_{2i+1} = x_{2i} + \frac{9-\sqrt{15}}{6}k,\ i=0,1,\dots,n\},$$

where $k^{-1} = n + \frac{9-\sqrt{15}}{6}$, $n \in \mathbb{N}$.

Using (6) we obtain the discrete analogue to (BVP2):

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$$\text{(DBVP2)} \qquad (T_k z)_i = \begin{cases} \beta_o, & i = 0, \\ f_i^k, & i = 1,2,\ldots,2n, \\ \beta_1, & i = 2n+1, \end{cases}$$

where

$$(T_k z)_i = \begin{cases} z_i, & i=0,2n+1, \\ k^{-2}\left[Az_{i-1}+Bz_{i+1}+Cz_{i+3}+Dz_{i+5}\right]-q_i z_i, & i=2m+1,\ m=0,1,\ldots, \\ k^{-2}\left[Dz_{i-5}+Cz_{i-3}+Bz_{i-1}+Az_{i+1}\right]-q_i z_i, & i=2m,\ m=3,4,\ldots,n, \\ k^{-2}\left[A_1 z_o+B_1 z_1+C_1 z_3\right]-q_2 z_2, & i=2, \\ k^{-2}\left[C_1 z_{2n-2}+B_1 z_{2n}+A_1 z_{2n+1}\right]-q_{2n-1} z_{2n-1}, & i=2n-1 \\ k^{-2}\left[A_2 z_1+B_2 z_3+C_2 z_5\right]-q_4 z_4, & i=4, \\ k^{-2}\left[C_2 z_{2n-4}+B_2 z_{2n-2}+A_2 z_{2n}\right]-q_{2n-3} z_{2n-3}, & i=2n-3, \end{cases}$$

and

$$A_1 = -\frac{414+2\sqrt{15}}{132}, \quad B_1 = \frac{623-20\sqrt{15}}{132}, \quad C_1 = -\frac{19-2\sqrt{15}}{12},$$

$$A_2 = -\frac{15\sqrt{15}-8}{12}, \quad B_2 = -2A_2, \quad C_2 = A_2,$$

$$f^k = \left[\beta_o, f_1, f_2, \ldots, f_{2n}, \beta_1\right]^T, \quad f_i = f(x_i), \quad q_i = q(x_i), \quad x_i \in J_k,$$

$$z = \left[z_o, z_1, \ldots, z_{2n+1}\right]^T.$$

The discrete problem to (BVP2) we write in the form

$$\text{(7)} \qquad B_k z = f_k,$$

where

$$\text{(8)} \quad B_k = k^{-2} \begin{bmatrix} k^2 & & & & & & & & & & \\ A_1 & B_1 & -k^2q_2 & C_1 & & & & & & & \\ A_1 & -k^2q_1 & B & 0 & C & 0 & D & & & 0 & \\ & A_2 & 0 & B_2 & -k^2q_4 & C_2 & & & & & \\ & & A & -k^2q_3 & B & 0 & C & 0 & D & & \\ & D' & 0 & C & 0 & B & -k^2q_6 & A & & & \\ & & & & A & -k^2q_5 & B & 0 & C & 0 & D \\ & & & D & 0 & C & 0 & B & -k^2q_8 & A & \\ & & & & & & A & -k^2q_{2n-5} & B & 0 & C \\ & & 0 & & & D & 0 & C & 0 & B & -k^2q_{2n-2} \\ & & & & & & & C_2 & -k^2q_{2n-3} & B_2 & 0 \\ & & & & & & D & 0 & C & 0 & B \\ & & & & & & & & & C_1 & -k^2q_{2n-1} \end{bmatrix} \begin{matrix} A_2 \\ A \\ D \\ 0 \\ A \\ B_1 \end{matrix}$$

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(9) $f_k = [\beta_0, f_2, f_1, f_4, f_3, \ldots, f_{2n-3}, f_{2n}, f_{2n-1}, \beta_1]^T$.

3. PROPERTIES OF THE SCHEMES

In this section we shall considere the discrete analogues (DBVP1) and (DBVP2) to (BVP1) and (BVP2). We shall prove that (DBVP1) and (DBVP2) have unique solutions, say y and z, for which we have

$$\|u^h - y\|_\infty \leq M_1 h^3, \quad \|u^k - z\|_\infty \leq M_2 k^3 ,$$

where u^h and u^k denote the restriction of the exact solution of (BVP1) and (BVP2) to the mesh I_h and J_k, and M_1, M_2 are the constants indepedent of h and k.

We shall begin with some notations (see |1|,|2|,|7|,|8|). For $x,y \in \mathbb{R}^m$, we write

$$x \leq (<) y \quad \text{iff} \quad x_i \leq (<) y_i , \quad i=1,2,\ldots,m,$$
$$|x| = [|x_1|, |x_2|, \ldots, |x_m|]^T .$$

Any $e \in \mathbb{R}^m$, $e > 0$, defines the norm $\|x\|_e = \max\limits_{1\leq i\leq m} \frac{|x_i|}{e_i}$ on $\mathbb{R}^m$. In particular $e = [1,1,\ldots,1]^T$ yields the maximum norm $\|\ \|_\infty$.

A mapping F of $\mathbb{R}^m$ into itself is called monotone if $x \leq y \Rightarrow Fx \leq Fy$ for any $x,y \in \mathbb{R}^m$.

For mappings F,G of $\mathbb{R}^m$ into itself we write $F \leq G$ iff G - F is monotone.

The set of matrices of the format m x m is denoted by $\mathbb{R}^{m,m}$.

For any $A = [a_{ij}] \in \mathbb{R}^{m,m}$ the matrices A_d, A_a, $A_a^- \in \mathbb{R}^{m,m}$ are defined via $A_d = \mathrm{diag}(a_{11}, a_{22}, \ldots, a_{mm})$, $A_a = A - A_d$, $A_a^- = [a_{ij}^-]$,

$$a_{ij}^- = \begin{cases} a_{ij} & \text{if } a_{ij} < 0, \\ 0 & \text{if } a_{ij} \geq 0. \end{cases}$$

Let $\tau^0(x) = \{i: i=1,2,\ldots,m, x_i = 0\}$, $\tau^+(x) = \{i: i=1,2,\ldots,m, x_i > 0\}$ for $x \in \mathbb{R}^m$.

If τ^1 and τ^2 are disjoint subsets of $\{1,2,\ldots,m\}$ we

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say that $A=[a_{ij}]\in \mathbb{R}^{m,m}$ connects τ^1 with τ^2 if for all $i\in\tau^1$ there are point $i_o=i,i_1,i_2,\dots,i_r\in\{1,2,\dots,m\}$ such that $a_{i_{j-1}i_j}\neq 0$, $j=1,2,\dots,r$ and $i_r\in\tau^2$.

The matrix A is called an L-matrix if $A_a\leq 0$.

$A\in\mathbb{R}^{m,m}$ is called an inverse-monotone matrix if A has an inverse $A^{-1}\geq 0$. Throughout this paper we shall use the abbreviation i.m. for inverse-monotone. The inverse-monotone L-matrix is called a M-matrix.

The following 5 theorems and their proofs can be found in |2|,|3|,|6|,|7|.

THEOREM 1. *Let* $A\in\mathbb{R}^{m,m}$, $\delta\in\mathbb{R}^m$. *If* $A_a\leq 0$, $\delta\geq 0$, $A\delta\geq 0$ *and if* A *connects* $\tau^o(A\delta)$ *with* $\tau^+(A\delta)$ *then* A *is an* M-*matrix.*

THEOREM 2. *Let* $A=[a_{ij}]\in\mathbb{R}^{m,m}$, $A_a^-=A^z+A^s$, $A^z=[a_{ij}^{(z)}]\leq 0$, $A^s=[a_{ij}^{(s)}]\leq 0$. *The matrix* A *is i.m. if the following conditions are satisfied:*

1. A_d+A^z *is an* M-*matrix,*
2. $a_{ij}\leq\sum_{k=1}^{m} a_{ik}^{(z)}a_{kk}^{-1}a_{kj}^{(s)}$ *for all* $a_{ij}>0$, $i\neq j$,
3. *there exists* $e\geq 0$ $(e\in\mathbb{R}^m)$ *such that* $Ae\geq 0$ *and* A^z *or* A^s *connects* $\tau^o(Ae)$ *with* $\tau^+(Ae)$.

THEOREM 3. *Let* $A\leq B$ $(A,B\in\mathbb{R}^{m,m})$ *and assume that* B *is i.m. Then* A *is i.m. iff there exists* $e>0$ $(e\in\mathbb{R}^m)$ *such that* $B^{-1}Ae>0$.

THEOREM 4. *Let* $C\leq A\leq B$ $(A,B,C\in\mathbb{R}^{m,m})$. *If* B *and* C *are i.m. then* A *is i.m.*

THEOREM 5. *Let* F *be a nonlinear mapping* $\mathbb{R}^m$ *into itself and let* $P,Q\in\mathbb{R}^{m,m}$ *be such that* $Q\leq P$, $Q\leq F\leq P$. *Let* $A,S\in\mathbb{R}^{m,m}$ *be such that* A-P, A-S *are i.m.,* $2S\leq P+Q$. *Then* $(A-F)^{-1}$ *exists and* $|(A-F)^{-1}-(A-F)^{-1}w|\leq(A-P)^{-1}|v-w|$, *for any* $v,w\in\mathbb{R}^m$. *Furthermore, the parallel chord iteration*

$$x^o\in\mathbb{R}^m,\quad (A-S)x^n=(F-S)x^{n-1},\quad n\in\mathbb{N}$$

converges for any $x^o\in\mathbb{R}^m$ *to the unique solution of* $Ax=Fx$.

Now we shall consider (DBVP1).

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THEOREM 6. *The matrix* A_h *from (5) is i.m. and for any matrix* $D = \mathrm{diag}(0, \mu_1, \mu_2, \ldots, \mu_{2n}, 0)$ *such that*

$$\mu_i \in [-h^{-2} q_o, \lambda_o), \quad i=1,2,\ldots,2n, \quad q_o = 3(\sqrt{5}-1), \quad 0 < \lambda_o < 8, \tag{10}$$

the matrix $A_h - D$ *is i.m.*

P r o o f. We can write the matrix A_h in the form

$$A_h = h^{-2} ML,$$

where

$$M = \begin{bmatrix} h^2 & & & & & & \\ m_1 & m_2 & & & & & \\ m_2 & m_1 & & & & & \\ & & m_1 & m_2 & & & \\ & & m_2 & m_1 & & & \\ & & & & \ddots & & \\ & & & & & m_1 & m_2 \\ & & & & & m_2 & m_1 \\ & & & & & & h^2 \end{bmatrix}, \quad L = \begin{bmatrix} 1 & & & & & & \\ \ell_2 & 1 & \ell_1 & & & & \\ & \ell_1 & 1 & \ell_2 & & & \\ & & \ell_2 & 1 & \ell_1 & & \\ & & & \ell_1 & 1 & \ell_2 & \\ & & & & \ddots & & \\ & & & & \ell_2 & 1 & \ell_1 \\ & & & & & \ell_1 & 1 & \ell_2 \\ & & & & & & & 1 \end{bmatrix}$$

$M, L \in \mathbb{R}^{2n+2, 2n+2}$, $m_1 = 1 + \frac{\sqrt{5}}{5}$, $m_2 = -2(1 - \frac{2\sqrt{5}}{5})$, $\ell_1 = -\frac{3-\sqrt{5}}{2}$, $\ell_2 = -\frac{\sqrt{5}-1}{2}$.

Since $\delta = [1,1,\ldots,1]^T > 0$, $M\delta = [h^2, \sqrt{5}-1, \ldots, \sqrt{5}-1, h^2]^T > 0$, Theorem 1 yields that M is i.m. Taking δ again, we have $L\delta = [1,0,0,\ldots,0,1]^T \geq 0$, $\tau^o(L\delta) = \{1,2,\ldots,2n\}$, $\tau^+(L\delta) = \{0, 2n+1\}$. The matrix L connects $\tau^o(L\delta)$ with $\tau^+(L\delta)$ (for $i_o \in \tau^o(L\delta)$ we define $i_{j+1} = i_j - 1$, $j=0,1,\ldots,r-1$, $i_r = 0$) and Theorem 1 yields that L is i.m. The matrix A_h is the product of two i.m. matrices M and L, i.e. A_h is an i.m. matrix.

Let $e \in \mathbb{R}^{2n+2}$ be defined by

$$e_i = x_i(1-x_i) + \xi, \quad x_i \in I_h, \quad \xi = \frac{8-\lambda_o}{4\lambda_o} > 0.$$

Now we have $e > 0$, $(A_h e)_o = (A_h e)_{2n+1} = \xi$, $(A_h e)_i = \lambda_i e_i$,

$i=1,2,\ldots,2n$, where

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$$\lambda_i = \frac{2}{x_i(1-x_i)+\xi} \geq \frac{2}{0.25+\xi} = \lambda_o, \quad i=1,2,\ldots,2n.$$

Let $D_1 = \mathrm{diag}(0,d_1,d_2,\ldots,d_{2n},0) \geq 0$ satisfy condition (10). Then $A_h - D_1 \leq A_h$, $(A_h-D_1)e > 0$. Since $A_h^{-1} \geq 0$ it follows $A_h^{-1}(A_h - D_1)e > 0$ and Theorem 3 yields that $A_h - D_1$ is i.m. Let $D_2 = \mathrm{diag}(0,d_1',d_2',\ldots,d_{2n}',0) \leq 0$ satisfy condition (10). Let $(A_h - D_2)_a = A^z + A^s$, where $A^z = [z_{ij}]$, $A^s = [s_{ij}]$ is defined via

$$z_{ij} = h^{-2}\begin{cases} -(3-\sqrt{5}) & \text{if } i=2m,\ j=2m-1,\ m=1,2,\ldots,n, \\ & \quad i=2m-1,\ j=2m,\ m=1,2,\ldots,n, \\ 0 & \text{otherwise,} \end{cases}$$

$$s_{ij} = h^{-2}\begin{cases} -\frac{2\sqrt{5}}{5} & \text{if } i=2m,\ j=2m+1,\ m=1,2,\ldots,n \\ & \quad i=2m+1,\ j=2m,\ m=0,1,\ldots,n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Then applying Theorem 1 to $\delta = [1,1,\ldots,1]^T$ we see that $(A_h - D_2)d + A^z$ is i.m. Since

$$d(b-h^2 d_i') \leq a \ c <=> d_i' \geq -h^{-2}q_o, \text{ and } (A_h-D_2)e \geq A_h e > 0,$$

Theorem 2 yields that $A_h - D_2$ is i.m. Now let the matrix $D = \mathrm{diag}(0,\mu_1,\mu_2,\ldots,\mu_{2n},0)$ satisfy condition (10). Then there exist matrices $D_1 \geq 0$ and $D_2 \geq 0$ such that $D_2 \leq D \leq D_1$. Since $A_h - D_1 \leq A_h - D \leq A_h - D_2$, Theorem 4 yields that $A_h - D$ is i.m.

THEOREM 7. *Suppose that conditions* (L), (1), (2) *are satisfied. Then for any* $v,w \in \mathbb{R}^{2n+2}$

$$|v-w| \leq (A_h - M)^{-1} |L_h v - L_h w|,$$

where $M = \mathrm{diag}(0,\mu,\mu,\ldots,\mu,0) \in \mathbb{R}^{2n+2,2n+2}$

THEOREM 8. *Let* $S = \mathrm{diag}(0,s_1,s_2,\ldots,s_{2n},0)$, $s_i \geq -h^{-2}q_o$, $i=1,2,\ldots,2n$; *and* $s = \max_{1\leq i\leq 2n} s_i \leq \frac{1}{2}(\mu+q)$. *Then the parallel chord iteration*

$$y^o \in \mathbb{R}^{2n+2}, \quad (A_h - S)y^m = (F_h - S)y^{m-1}, \quad m \in \mathbb{N}$$

converges for any y^o *to the unique solution of* $A_h y = F_h y$.

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The proofs of Theorem 7 and Theorem 8 follow directly from Theorem 5.

THEOREM 9. *Suppose that conditions* (L),(1),(2) *are satisfied. Then*

$$\|v-w\|_e \leq \frac{1}{K}\|L_h v - L_h w\|_e, \quad K = \min(\xi, \lambda_o - \mu), \text{ for any } v,w \in \mathbb{R}^{2n+2},$$

$$\|v-w\|_\infty \leq \frac{2}{\lambda_o \xi K}\|L_h v - L_h w\|_\infty, \text{ for any } v,w \in \mathbb{R}^{2n+2}.$$

P r o o f. Since $(A_h - M)e \geq Ke$ and $(A_h - M)^{-1} \geq 0$ it follows that

$$(A_h - M)^{-1} e \leq \frac{1}{K} e, \quad \|(A_h - M)^{-1}\|_e = \|(A_h - M)^{-1} e\|_e \leq \frac{1}{K}.$$

From Theorem 7 it follows that

$$\|v-w\|_e \leq \|(A_h - M)^{-1}\|_e \|L_h v - L_h w\|_e \leq \frac{1}{K}\|L_h v - L_h w\|_e \quad \text{for any}$$

$$v,w \in \mathbb{R}^{2n+2}.$$

Since

$$\frac{1}{0.25+\xi}\|z\|_\infty \leq \|z\|_e \leq \frac{1}{\xi}\|z\|_\infty \quad \text{for any } z \in \mathbb{R}^{2n+2},$$

we have

$$\|v-w\|_\infty \leq \frac{0.25+\xi}{\xi K}\|L_h v - L_h w\|_\infty, \text{ for any } v,w \in \mathbb{R}^{2n+2},$$

which completes the proof.

COROLLARY 1. *Suppose that conditions* (L),(1),(2) *are satisfied. Let* y *be the solution of* (DBVP1), u *the solution of* (BVP1) *and the vector* u^h *be the rectriction of* u *to the mesh* I_h. *Then*

$$u \in C^5[0,1] \Rightarrow \|y - u^h\|_\infty \leq M_1 h^3$$

where M_1 *is a constant independent of* h.

P r o o f. In |5| it is proved that $u \in C^5[0,1]$ implies that

$$L_h u^h - L_h y = O(h^3).$$

Assuming $v = u^h$ and $w = y$, from Theorem 9, there, follows the proof.

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We shall consider now the boundary value problem (BVP2) and its discrete analogue $B_k z = f_k$.

THEOREM 10. *Let condition* (3) *be satisfied. Then the matrix* B_k *is i.m.*

P r o o f. Let $(B_k)_a^- = B^z + B^s$, where $B^z = [b_{ij}^{(z)}]$,

$$b_{ij}^{(z)} = \begin{cases} \frac{C}{2} & \text{if } i=2m,\ j=2m+2,\ m=1,2,\dots,n-2, \\ & \quad i=2m+1,\ j=2m-1,\ m=2,3,\dots,n-1, \\ 0 & \text{otherwise.} \end{cases}$$

Since $(B_k)_d + B^z$ is an M-matrix, $2DB_2 \le A_2C$, $4BD \le C^2$ and

$$B_k e = [\xi, 2-q_2e_2,\ 2-q_1-e_1,\ 2-q_4e_4,\ 2-q_3e_3, \dots$$

$$\dots,\ 2-q_{2n}e_{2n},\ 2-q_{2n-1}e_{2n-1}, \xi]^T > 0, \quad \text{for}$$

$$e = [e_o, e_1, \dots, e_{2n+1}]^T > 0,\ e_i = x_i(1-x_i) + \xi,\ x_i \in J_k,$$

where $\xi > 0$ is determined so that $\mu < \frac{2}{0.25+\xi} < 8$, we conclude applying Theorem 2, that B_k is i.m.

THEOREM 11. *Suppose that condition* (3) *is satisfied. Let* z *be the solution of* (DBVP2), u *the solution of* (BVP2) *and the vector* u^k *the restriction of* u *to the mesh* J_k. *Then*

$$u \in C^5[0,1] \Rightarrow \| u^k - z \|_\infty \le M_2 k^3,$$

where M_2 *is a constant indepedent of* k.

P r o o f. Let $\varepsilon \in (0, \min(-A_1, \frac{A_1A_2}{2B_1-A_2}))$. Now multiply the zeroth row of the matrix B_k by εk^{-2} and add the result to the first and third row. Also, multiply the (2n+1)th row of matrix B_k by εk^{-2} and add the result to the (2n)th and (2n-2)th row. Now multiply the first, third, (2n)th and (2n-2)th row of matrix B_k by k^2. As a result one obtains a new matrix $\tilde{B}_k$. The equations

$$B_k z = f_k \text{ and } \tilde{B}_k z = \tilde{f}_k, \text{ with}$$

$$\tilde{f}_k = [\beta_o, k^2 f_2 + \varepsilon\beta_o, f_1, k^2 f_4 + \varepsilon\beta_o, f_3 f_6 f_5, \dots, f_{2n-2}, k^2 f_{2n-3} +$$

$$+ \varepsilon\beta_1, f_{2n}, k^2 f_{2n-1} + \varepsilon\beta_1, \beta_1]^T$$

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are equivalent. Using Theorem 2, with vector e from Theorem 10, we obtain that $\tilde{B}_k$ is i.m.

Let $\lambda = \min(1, \frac{\varepsilon\xi}{0.25+\xi}, \frac{2}{0.25+\xi} - \mu) > 0$. Then $\tilde{B}_k e \geq \lambda e$ and $\| \tilde{B}_k^{-1} \|_e \leq \frac{1}{\lambda}$.

Since $\tau(u) = \tilde{B}_k u^k - \tilde{B}_k z = \tilde{B}_k u^k - \tilde{f}_k = O(k^3)$ and $u^k - z = \tilde{B}_k^{-1} \tau(u)$, we have

$$\| u^k - z \|_\infty \leq (0.25+\xi) \| u^k - z \|_e \leq (0.25+\xi) \| B_k^{-1} \|_e \| \tau(u) \|_e \leq$$

$$\leq (0.25+\xi) \frac{1}{\lambda} \cdot \frac{1}{\xi} \| \tau(u) \|_\infty \leq M_2 k^3 .$$

4. NUMERICAL RESULT

In this section we shall present a numerical example :

$$-u'' = -e^u , \quad x \in [0,1], \quad u(0) = u(1) = 0.$$

This problem is considered in |3| and |6|, and its exact solution is

$$u(x) = -\ln 2 + 2 \ln(c \sec \frac{c(x-0.5)}{2}) , \quad c = 1.336055\ldots .$$

For $x \in [0,1]$ is $-1 \leq u(x) \leq 0$.

To compute the approximation of $u(x)$ we define (see |2|)

$$f(x,v) = \begin{cases} -e^{-1} , & \text{for } v \leq -1, \\ -e^v , & \text{for } -1 \leq v \leq 0 , \\ -1 , & \text{for } v \geq 0, \end{cases}$$

satisfying (L) with $q = e^{-1} - 1$, $\mu = 0$. Using our first scheme, with $n = 20$, we iterate according to

$$y^0 = [1,1,\ldots,1]^T \in \mathbb{R}^{42}, \quad A_h y^m = F_h y^{m-1}, \quad m \in \mathbb{N} .$$

From Theorem 7 (see |2|) it follows that

$$|y^m - y| \leq A_h^{-1} |A_h y^m - F_h y^m|, \quad m \in \mathbb{N},$$

where y is the solution of (DBVP1) for our example. Also it holds that

$$|y^m - y| \leq w^m , \quad m \in \mathbb{N} ,$$

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where w^m is a solution of $A_h w^m \geq |A_h y^m - F_h y^m|$, $m \in \mathbb{N}$.

We have calculated in double precision arithmetic on PDP 11/340.

The numerical results are

$$h = 0.18740391 \cdot 10^{-1},\ m = 16,\ \|y^m - y\|_\infty < 10^{-16},$$

$$\|y^m - u^h\|_\infty \leq 10^{-7},$$

where u^h denotes the restriction of u to the mesh I_h.

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REZIME

O NUMERIČKOM REŠAVANJU KONTURNOG PROBLEMA
POMOĆU OPTIMALNOG NUMERIČKOG DIFERENCIRANJA

U radu se posmatra numeričko rešavanje konturnih problema (BVP1) i (BVP2) pod pretpostavkama (L), (1), (2) osnosno (3). Za formiranje diskretnih analogona (DBVP1) i (DBVP2) koriste se optimalne četvorotačkaste formule reda 3 za numeričku aproksimaciju drugog izvoda iz |4| i |5|. Pri tom se koriste specijalne mreže diskretizacije I_h i J_k. Dokazano je da pod navedenim pretpostavkama opisani diferencni postupci imaju red konvergencije 3 i da postupak paralelne sečice za rešavanje (DBVP1) konvergira. Numerički je rešen primer iz |3| i |6|.

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ON THE CONVERGENCE OF SOME FINITE DIFFERENCE SCHEMES FOR A SINGULAR PERTURBATION PROBLEM

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ABSTRACT

A difference schemes was formed with a constant fitting factor on a nonuniform mesh for a singular perturbation problem of the second order and for boundary conditions of the third order. A uniform convergence of the order one was proved with respect to the small parametar of the solution and its first derivative. Emelyanov's technique was used.

We shall consider a problem of the form

$$Lu(x) = \varepsilon u''(x) + a(x)u'(x) + b(x)u(x) = f(x) \tag{1}$$
$$u'(0) - \beta u(0) = \mu_1 , \quad u(1) = \mu_2$$

where $\beta \geq 0$ and the functions a,b and f are sufficiently smooth and satisfy $a(x) \geq \alpha > 0$, $b(x) \leq 0$.

By M and M_i we denote the constants which are independent on ε and h (h is determined by (3)).

Emelyanov [2] presented a difference scheme on a uniform mesh for a solving problem (1) and postulated that it is uniformly convergent with the order one. The first derivative which is obtained from scheme [2] has the order one of the uniformly convergence with respect to ε. The result of [7] represent a generalization of the results of [2] on a nonuniform mesh. In

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this paper we shall show that the fitting factor in [2] and [7], which depends of $a(x_i)$, can be replaced with either constant or "quasi constant" one [4], which depends only on $a(0)=\omega$. In [2] and [7] for obtaining an estimation of the difference between the numerical and asymptotic solution the appearing expressions $\exp[\pm(a(x_i)-a(0))h/\varepsilon]$ create difficulties in a non classical error estimation.

Taking into account the results of [1] and the asymptotic expansions [8] we use an exponentially fitting factor of the form

$$\sigma_i(\rho,\omega) = \omega \frac{x_{i+1}-x_i}{\varepsilon} \operatorname{cth} \omega \frac{x_{i+1}-x_i}{\varepsilon} .$$

Then the schemes in [2] and [7] become simpler and the conditions of the grid in [7] are relaxed. At the end we shall present some numerical experiments. Further we shall compare the schemes from [2] and [7] and this one. We shall also compare a uniform mesh and a nonuniform one and present a numerical validation of the theoretical order of the uniform convergence for the above schemes. The experimental determination of the order of uniform convergence is based on the General Convergence Principle [1].

Problem (1) can be written in the form

$$(2) \qquad \begin{aligned} L_1(u,v) &= \varepsilon v' + av + bu = f \\ L_2(u,v) &= u' - v = 0 \end{aligned}$$

$$v(0) - \beta u(0) = \mu_1, \quad u(1) = \mu_2 .$$

For solving system (2) we use a nonuniform mesh

$$(3) \qquad \begin{aligned} &x_{i+1} = x_i + \sigma_i h, \quad -\eta_{i+1} = \sigma_i, \quad \sigma_i \le M, \quad h^{-1} = -\sum_{i=1}^{n} \eta_i \\ &i = \overline{1,n-1}, \quad x_o = 0, \quad x_1 = 1, \quad \sigma_1 = -\eta_1 = 1, \quad \sigma_i \in \mathbb{R} \\ &|\sigma_i + \eta_i| \le M_i x_i , \quad x_i \le C_o, \quad C_o \text{ is given, } 0 < C_o \le 1. \end{aligned}$$

and

$$(4) \qquad \begin{aligned} L_1^h(u_o,v_o) &= (\varepsilon\omega\rho/2) \operatorname{cth}(\omega\rho/2)(v_1-v_o)/2 + \\ &+ b(h/2)(u_I+u_o)/2 = f(h/2) \quad \rho = h/\varepsilon . \end{aligned}$$

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$$L_2^h(u_o,v_o) = (u_1-u_o)h - (v_1-v_o)/2 = 0$$

$$L_1^h(u_i,v_i) = (\sigma_i(\rho,\omega)/\rho)(\bar{A}_i v_{i-1} + \bar{B}_i v_i + \bar{c}_i v_{i+1}) +$$

(4) $$+ (\bar{\alpha}_i v_{i-1} + \bar{\beta}_i v_{i+1}) a(x_i) + b(x_i)(\bar{\alpha}_i u_{i-1} + \bar{\beta}_i u_{i+1}) = f(x_i)$$

$$L_2^h(u_i,v_i) = (\bar{A}_i u_{i-1} + \bar{B}_i u_i + \bar{c}_i u_{i+1})/h - (\bar{\alpha}_i v_{i-1} + \beta_i v_{i+1}) = g_i$$

$$(i = \overline{1,n-1})$$

where

$$\sigma_i(\rho,\omega) = \Omega_i \operatorname{cth}\Omega_i, \quad \Omega_i = \omega\sigma_i\rho$$

$$\bar{A}_i = \sigma_i/\eta_i(\sigma_i-\eta_i) \quad \bar{B}_i = -(\eta_i+\sigma_i)/\sigma_i\eta_i, \quad \bar{C}_i = \eta_i/\sigma_i(\eta_i-\sigma_i)$$

$$\bar{\alpha}_i = \sigma_i/(\sigma_i-\eta_i), \quad \bar{\beta}_i = -\eta_i/(\sigma_i-\eta_i), \quad g_i = 0.$$

We use the solution of system (4) for an approximation of the exact values $u(x)$ and $u'(x)$ in the points of the grid

$$u'(x_i) \approx v_i, \quad u(x_i) \approx u_i \quad (i=0,n)$$

It can be shown that a solution of system (4) can be obtained from the recurrent relations

(5) $$u_{i+1} = t_i u_o + s_i$$
$$v_{i+1} = p_i u_o + q_i, \quad (i=\overline{0,n-1})$$

similar to those in |7|.

Here

$$t_o = K_o[1 + (B_o + \beta + A_o\beta)h/2]$$
$$s_o = K_o h[(1 + A_o)\mu_1 + r_o + 2g_o]$$
$$p_o = K_o[(A_o + B_o h/2)\mu_1 + r_o + hB_o g_o]$$
$$K_o = (1 - B_o h/2)^{-1}$$
$$A_o = (\omega\operatorname{cth}(\omega\rho/2) - a(h/2))/\psi, \quad B_o = -b(h/2)/\psi$$
$$r_o = 2f(h/2)/\psi, \quad \psi = \omega\operatorname{cth}(\omega\rho/2) + a(h/2)$$
$$t_i = K_i(R_i p_{i-2} + S_i t_{i-2} + B_i\beta_i p_{i-1} + \delta_i t_{i-1})$$
$$s_i = K_i(R_i q_{i-2} + S_i s_{i-2} + \beta_i B_i q_{i-1} + \delta_i s_{i-1} + N_i)$$
$$p_i = K_i(F_i p_{i-2} + Q_i t_{i-2} + B_i p_{i-1} + \delta_i D_i t_{i-1})$$
$$q_i = K_i(F_i q_{i-2} + Q_i s_{i-2} + B_i q_{i-1} + \delta_i D_i s_{i-1} + M_i)$$
$$(i=1,2,\ldots,n-1)$$

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$$p_{-1} = \beta, \quad q_{-1} = \mu_1, \quad t_{-1} = 1, \quad s_{-1} = 0$$

$$K_i = (1-\beta_i D_i)^{-1}$$

$$R_i = \beta_i A_i + \alpha_i, \qquad Q_i = \gamma_i D_i + C_i$$

$$S_i = C_i \beta_i + \gamma_i, \qquad M_i = r_i + \rho_i D_i$$

$$F_i = \alpha_i D_i + C_i, \qquad N_i = \rho_i + r_i \beta_i$$

$$A_i = \sigma_i(\omega\sigma_i \operatorname{cth}\Omega_i + a_i\eta_i)/\eta_i^2\phi, \quad a_i = a(x_i)$$

$$\phi = \omega \operatorname{cth}\Omega_i + a_i$$

$$B_i = -(\sigma_i^2-\eta_i^2)\omega \operatorname{cth}\Omega_i/\eta_i^2\phi, \quad C_i = b_i\sigma_i/\eta_i\phi$$

$$D_i = -b_i/\phi, \quad r_i = f_i(\sigma_i-\eta_i)/\eta_i\phi$$

$$\alpha_i = -h\sigma_i^2/\eta_i, \quad \beta_i = h\sigma_i, \quad \gamma_i = \sigma_i^2/\eta_i^2$$

$$\delta_i = (\eta_i^2-\delta_i^2)/\eta_i^2, \quad \rho_i = hg_i .$$

LEMMA 1. *Let* u_i *and* v_i *be a solution of system (4). Let* $|K_i| \le M$, $|t_{n-1}^{-1}| \le M$. *Then*

$$|u_i| + |v_i| \le M\,(|\mu_1| + |\mu_2| + H), \text{ where } H \text{ is determined by}$$

$$|f(x_i)|;\ h|g_i| \le H \quad (i=0,1,\ldots,n).$$

The proof is completely analogous to that of Lemma 1 [7].

THEOREM 1. *Let* u_i *and* v_i *be a solution of system (4). Then the following estimation holds*

$$|u_i - u(x_i)| + |v_i - u'(x_i)| \le M\left(\frac{h^2}{\varepsilon} + \frac{h^2}{\varepsilon^2} + \frac{h^4}{\varepsilon^4}\right).$$

The proof is obtained by using the techniques outlined in [7].

For a "non classical" convergence we use the asymptotic solution of problem (1) which has the form ([8]):

$$u(x) = \psi_o(x) + \varepsilon\psi_1(x) + \varepsilon u_o(x/\varepsilon) + \varepsilon^2 u_1(x/\varepsilon) + w(x)$$

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$$v(x) = \phi_o(x) + \varepsilon\phi_1(x) + v_o(x/\varepsilon) + \varepsilon v_1(x/\varepsilon) + z(x)$$

where ϕ_i and ψ_i are the solutions of the differential equations system, independent on ε .

$$u_o(t) = -\frac{\mu}{\omega} e^{-\omega t}, \quad v_1(t) = \mu e^{-\omega t}$$

$$u_1(t) = p_1(t)e^{-\omega t}, \quad v_1(t) = p_2(t)e^{-\omega t},$$

$p_i(t)$ are polynomials of the order two. According to [2] the following estimations hold:

$$|w(x)| + |z(x)| \leq M\varepsilon^2$$

$$|v^{(k-1)}| + |u^{(k)}| \leq M\varepsilon^{k-1} \qquad (k = 1,2,3,4).$$

We now introduce the notations

$$Tu = \bar{\alpha}_i u_{i-1} + \bar{\beta}_i u_{i+1}$$

$$Du = h^{-1}(\bar{A}_i u_{i-1} + \bar{B}_i u_i + \bar{C}_i u_{i+1})$$

$$w_i^h = u(x_i) - u_i - w(x_i)$$

$$z_i^h = v(x_i) - v_i - z(x_i), \quad t_i = x_i/\varepsilon .$$

By applying the operators L_1^h and L_2^h to the mesh functions $w_i^h(x_i)$ and $z_i^h(x_i)$ we obtain

$$L_1^h(z_i^h, w_i^h) = \sum_{k=1}^{5} \Phi_{ki}, \quad L_2^h(z_i^h, w_i^h) = \sum_{k=1}^{3} H_{ki}$$

$$L^h(1) = -\varepsilon u_o(1/\varepsilon) - \varepsilon^2 u_1(1/\varepsilon)$$

$$z^h(0) - \beta w^h(0) = -\beta y_o \varepsilon^2 \quad , \text{ where}$$

$$\Phi_{1i} = -(-f_i + a_i T\phi_o + b_i T\psi_o + \varepsilon(\sigma_i(\rho)-1)D\phi_o + \varepsilon D\phi_o +$$

$$+ \varepsilon a_i T\phi_1 + \varepsilon b_i T\psi_1 + \varepsilon^2(\sigma_i(\rho)D\phi_1 + b_i T u_1 t_i)$$

$$\Phi_{2i} = \varepsilon x_i a'(\xi)\bar{\beta}_i p_2(t_{i+1})e^{-\omega t_{i+1}}$$

$$\Phi_{4i} = -x_o\varepsilon\left(\frac{\sigma_i(\rho,\omega)}{\rho}\bar{A}_i e^{-\omega t_{i-1}} + \frac{\sigma_i(\rho,\omega)}{\rho}\bar{B}_i e^{-\omega t_i} - a_i\bar{\alpha}_i e^{-\omega t_{i-1}} + \right.$$

$$\left. + \omega\frac{1}{\mathrm{sh}\Omega_i}\frac{\eta_i}{\eta_i - \sigma_i} e^{-\omega t_i}\right)$$

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$$\Phi_{5i} = A + \frac{\omega_1}{2} B \quad .$$

$$A = -\omega\sigma_i \mu \operatorname{cth}\Omega_i h D e^{-\omega t_i} - \omega\mu T e^{\omega t_i} - a'(\varrho) x_i T e^{\omega t_i}$$

$$B = \varepsilon\mu \left(\bar{A}_i \frac{\sigma_i(\rho,\omega)}{\rho} e^{-\omega t_{i-1}} t_{i-1}^2 + \bar{B}_i \frac{\sigma_i(\rho,\omega)}{\rho} t_i^2 e^{\omega t_i} + \right.$$

$$\left. + \omega t_{i-1}^2 \frac{e^{-\omega t_i} \eta_i}{\operatorname{sh}\, \Omega_i(\eta_i - \sigma_i)} + a_i t_{i-1}^2 e^{-\omega t_{i-1}}\right)$$

$$H_{1i} = -(D\psi_o - T\phi_o) - \varepsilon(D\psi_1 - T\phi_1)$$

$$H_{2i} = -\varepsilon \frac{\mu}{\omega} D e^t + \mu T e^{-\omega t}$$

$$H_{3i} = \varepsilon^2 D u_1(t) - \varepsilon T v_1(t) \, .$$

Using the Taylor expansions of the above functions about x_i and the following inequalities ([6], [1]):

$$c' e^t \leq \operatorname{sht} \leq c'' e^t \, , \qquad 0 < c < t < \infty$$

$$c' t \leq \operatorname{sht} \leq c'' t \qquad 0 < t < c$$

c, c' and c" are constants.

$$t^j e^{-t} \leq c(\theta) e^{-\theta t}, \quad \theta \in (0,1), \quad 0 \leq t < \infty$$

we can prove Lemma 2.

LEMMA 2. *For* Φ_{ki} *and* H_{ki} *the following estimations hold*

$$|\Phi_{ki}| \leq M(\varepsilon^2 + h)$$

$$|H_{ki}| \leq M(\varepsilon^2 + h).$$

We now turn to the "non classical" estimation contained in Theorem 2.

THEOREM 2. *Let* u_i *and* v_i *be a solution of system (4) on grid (3). Then the following estimation holds*

$$|u_i - v(x_i)| + |v_i - u'(x_i)| \leq M(h + \varepsilon^2) \, .$$

The proof is obtained by applying Lemma 1 and Lemma 2.

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THEOREM 3. *Let* u_i *and* v_i *be the solution of system (4) on grid (3). Then the estimation*

$$|u_i - u(x_i)| + |v_i - u'(x_i)| \leq Mh \quad , \text{ holds.}$$

Using Theorem 1 when $h \leq \varepsilon^2$ and Theorem 2 when $h \geq \varepsilon^2$ the proof is immediate.

For choosing η_i, σ_i and c_o in (3) we use the test for the order of uniform convergence [1]. The following theorem transforms the results of [1] on a nonuniform mesh.

THEOREM 4. *Let* F^{m_k} *be the mesh function defined on the mesh*

$$I_{m_k} = \{x_i, x_i = x_{i-1} + \sigma_i h, (i = 1, \ldots, m_k - 1),\ x_o = 1,\ h^{-1} = \sum_{i=1}^{m_k} \sigma_i\}$$

and p *be a positive number. Then*

$$|F^{m_k} - F| \leq M_1 m_k^{-p} \qquad \text{for all} \quad m_k \geq m_o$$

if and only if the following two conditions hold:

a) $|F^{m_k} - F| = o(1), \quad as \quad m_k \to \infty$

b) $|F^{m_k} - F^{m_{k+1}}| \leq M_2 m_k^{-p} \quad for\ all \quad m_k \geq m_o$

$m_{k+1} = \lambda m_k, \quad \lambda > 1.$

The proof is similar to that [2] p. 20.

By applying Theorem 3 we form the test for the experimental determination of the order of uniform convergence for difference schemes on a nonuniform mesh. Let $m_{k+1} = 2m_k$, $I_{m_k} \subset I_{m_{k+1}}$, $u_j^{m_k}$ be an approximative solution and $u(x_j)$ the exact solution at the point x_j. We de note

$$z_{k,\varepsilon} = \max_j |u_j^{m_k} - u_j^{m_{k+1}}| \quad (k = 0, 1, \ldots)$$

where the maximum is taken over all the points of the coarser one of the two meshes.

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If the difference scheme is uniformly convergent, then

$$|u_j^{m_k} - u(x_j)| \leq M_1 m_k^{-p}$$

by applying Theorem 3 we can show that

$$|u_j^{m_k} - u(x_j)| \to o(1), \quad m_k \to \infty$$

$z_{k,\varepsilon} \leq M_2 m_k^{-p}$ then holds, and that

M_1 is independent of ε if and only if M_2 is. We take the quantities

$$\frac{z_{k,\varepsilon}}{z_{k+1,\varepsilon}} \simeq \frac{m_k^{-p}}{(2m_k)^{-p}} = 2^p \quad , \text{ and determine p in this way}$$

$$p = \log_2 (z_{k,\varepsilon}/z_{k+1,\varepsilon})$$

Numerical results. We treat problem |1.|

$$\varepsilon u''(x) + (1+x^2)u'(x) - (x - \tfrac{1}{2})^2 u(x) = 4(3x^2 - 3x + 1)(x+1)^2$$

$$u(0) - u'(0) = 0, \quad u(1) = 0$$

$$\varepsilon = 0.0625,$$

In the following series of figures and tables we present the results of our computations.

We use the notations

Scheme I ... $\sigma_i(\rho, \omega)$ with a uniform mesh

Scheme II ... $\sigma_i(\rho, a(x_i))$, with a uniform mesh

Scheme III ... $\sigma_i(\rho, a(x_i))$ mesh II

Scheme IV ... $\sigma_i(\rho, \omega)$ with mesh I:

Mesh I ... 0, 0.01, 0.02, 0.03, 0.04, 0.05, 0.07, 0.09, 0.11,
0.13, 0.16, 0.20, 0.24, 0.08, 0.32, 0.36, 0.40,
0.46, 0.52, 0.58, 0.64, 0.70, 0.75, 0.80, 0.84,
0.88, 0.92, 0.95, 0.97, 0.98, 0.99, 1.00.

Mesh II ... 0.0,0.02,0.04,0.0608,0.082368,0.1046733
0.1276864, 0.1513789, 0.1750714, 0.1997785
0.225500, 0.2522360, 0.2799866, 0.3087516,
0.3385311, 0.3693251, 0.4011336, 0.4339656,

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0.467794, 0.50266460, 0.5385124, 0.5753934
0.6132888, 0.6521987, 0.6921232, 0.7330621
0.7750155, 0.8179834, 0.8619658, 0.90699628
0.9529741, 1.00

For computing p we start with $m_o = 32$ and we put $m_{k+1} = 2m_k$ (k=0,1,2,3).

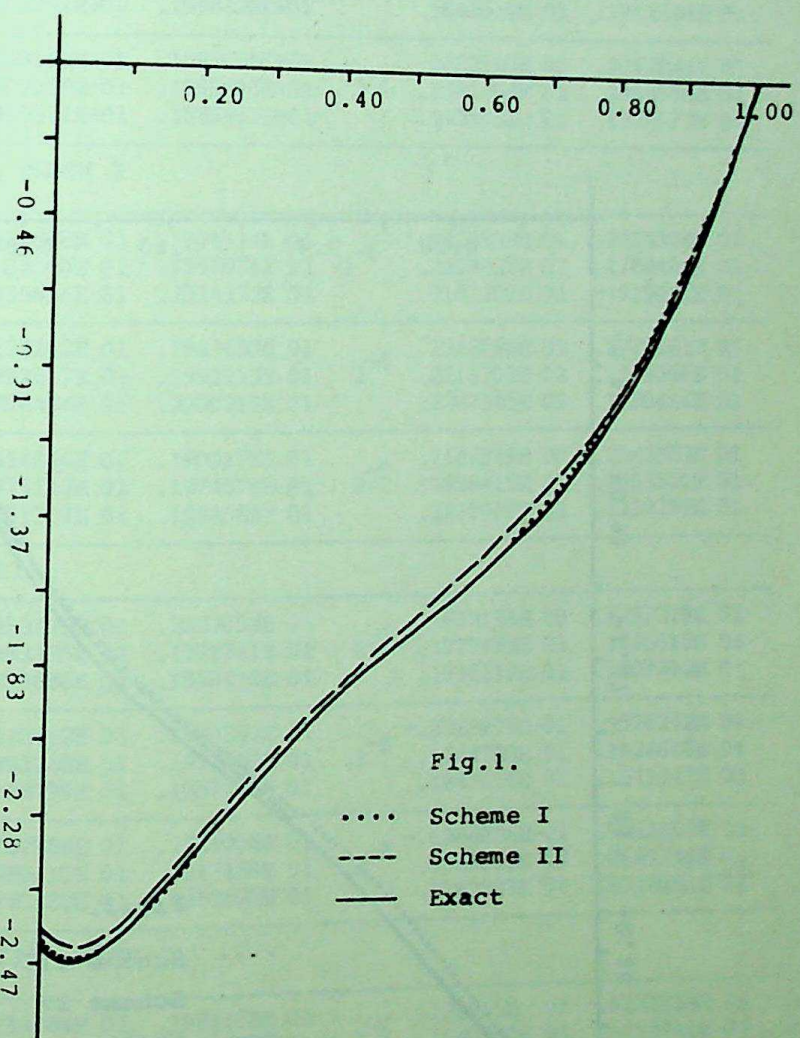

Fig.1.

.... Scheme I
---- Scheme II
—— Exact

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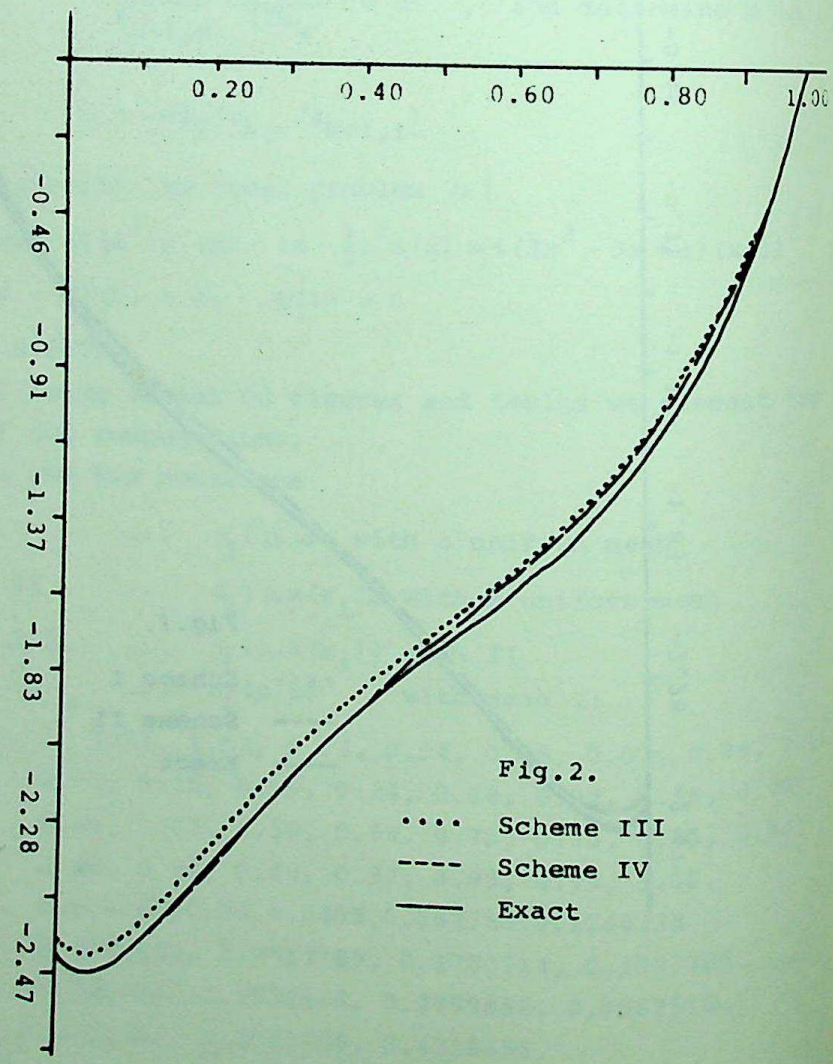

Fig.2.

.... Scheme III

---- Scheme IV

—— Exact

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TABELE 1

SCHEME I

ε	k	P_k	P_v	ε	P_u	P_v
2^{-1}	0	.1913569E+01	.2000908E+01	2^{-6}	.1059774E 01	.1729662E 01
	1	.1978778E+01	.1998136E+01		.190528 E 01	.195587 E 01
	2	.1982236E+01	.2010407E+01		.1979763E 01	.1976815E 01
2^{-3}	0	.1715132E+01	.1997260E+01	2^{-8}	.3478056E 00	.1105804E 01
	1	.2006710E+01	.1999607E+01		.1410312E 01	.1338642E 01
	2	.1922706E+01	.1998428E+01		.1842019E 01	.1711334E 01
2^{-4}	0	.1448952E+01	.1881095E+01	2^{-9}	.2882550E 00	.1062844E 01
	1	.2007009E+01	.1995805E+01		.1041304E 01	.1068558E 01
	2	.1895981E+01	.1886913E+01		.1420031E 01	.1329972E 01

SCHEME IV, MESH I

2^{-1}	0	.1926638E 01	.1996914E 01	2^{-6}	.1422633E 01	.1413524E 01
	1	.2004060E 01	.1998076E 01		.2208937E 01	.1784494E 01
	2	.1935634E 01	.2004372E 01		.2183209E 01	.1919046E 01
2^{-3}	0	.2124112E 01	.1963680E 01	2^{-8}	.1218086E 01	.1779681E 01
	1	.1997517E 01	.1992213E 01		.2218703E 01	.1158596E 01
	2	.1706424E 01	.2000217E 01		.2287345E 01	.1480441E 01
2^{-4}	0	.2414846E 01	.1900175E 01	2^{-9}	.7186019E 00	.2143383E 01
	1	.1973113E 01	.1974578E 01		.1858415E 01	.1053521E 01
	2	.1871747E 01	.1994568E 01		.2217098E 01	.1129259E 01

SCHEME II

2^{-1}	0	.2953125E 01	.2027054E 01	2^{-6}	.7675155E 00	.1293319E 01
	1	.1893125E 01	.1976741E 01		.1777995E 01	.1625615E 01
	2	.1799886E 01	.1990415E 01		.1936217E 01	.1800102E 01
2^{-3}	0	.2110542E 01	.1940582E 01	2^{-8}	-.3318977E-01	.9576272E 01
	1	.1993218E 01	.1972463E 01		.1244720E 01	.1024618E 01
	2	.2018799E 01	.1987117E 01		.1847666E 01	.1311047E 01
2^{-4}	0	.1787984E 01	.1831006E 01	2^{-9}	-.6986086E-01	.9262857E 01
	1	.1980410E 01	.1917159E 01		.9985099E 00	.9649740E 01
	2	.1995338E 01	.1958870E 01		.1262807E 01	.1034051E 01

SCHEME III MESH II

2^{-1}	0	.2213184E 01	.1981158E 01	2^{-6}	.133265E 01	.9237934E 01
	1	.1884465E 01	.1992462E 01		.1540363E 01	.1210781E 01
	2	.1030667E 01	.1999636E 01		.1775866E 01	.1628408E 01
2^{-3}	0	.1774796E 01	.183530E 01	2^{-8}	.1508032E 01	.8960677E 00
	1	.1956434E 01	.1950804E 01		.1990338E 01	.9521322E 00
	2	.2006448E 01	.1987238E 01		.1719587E 01	.1003877E 01

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2^{-4}	0	.1418320E 01	.156033E 01	2^{-9}	.1067063E 01	.8662433E 00
	1	.1896836E 01	.1855237E 01		.1488140E 01	.8518415E 01
	2	.1966221E 01	.1960452E 01		.1999397E 01	.975875E 01

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REZIME

O KONVERGENCIJI NEKIH KONAČNIH
DIFERENCNIH ŠEMA ZA SINGULARNE PERTURBACIONE
PROBLEME

U radu je data jedna diferencna šema sa konstantnim "fitting" faktorom za rešavanje konturnih problema za diferencijalne jednačine drugog reda sa malim parametrom uz najveći izvod i konturne uslove trećeg reda. Dokazana je uniformna konvergencija po malom parametru za rešenje i njegov prvi izvod. Koristi se neekvidistantna mreža.

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AN EXPONENTIALLY FITTED SCHEME ON A NON-UNIFORM MESH*

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ABSTRACT

The discretisation (6) of problem (1) is constructed on a non-uniform mesh. An exponentially fitted scheme is used and the linear convergence uniform on a small perturbation parameter is proved. In the case of a uniform mesh, our scheme reduces to the well known scheme of |1|. Numerical results are obtained on a special mesh, which is similar to the one in |3|.

1. INTRODUCTION

In this paper we shall consider the problem

$$L_\varepsilon u(x) := -\varepsilon^2 u''(x) + b^2(x)u(x) = f(x), \quad x \in I = [0,1], \tag{1}$$
$$u(0) = U_0, \quad u(1) = U_1,$$

where $b(x) \geq \beta > 0$, $x \in I$, $0 < \varepsilon \leq \varepsilon_0$, $\beta, \varepsilon, \varepsilon_0, U_0, U_1 \in \mathbb{R}$ and $b, f \in C^1(I)$. The parameter ε is a small perturbation parameter.

It is well known that there exists a unique solution $u_\varepsilon \in C^3(I)$ to problem (1). Furthermore, we have (|1|, |3|):

$$|u^{(i)}(x)| \leq M\varepsilon^{-i}, \quad x \in I, \quad i=0,1,2,3 \tag{2}$$

Here, and thereafter, M will denote each positive constant independent of ε. In section 3, M will be independent of discre-

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tisation mesh as well. The dependence on ε will always be denoted by writing ε as a subscript.

The solution u_ε has, in general, two boundary layers near the endpoints of the interval I. Our purpose is to solve (1) numerically by the difference method on a non-unifirm mesh. The reason for using a non-uniform mesh is our aim to obtain more mesh points in the region of boundary layers, those width is $O(\varepsilon)$, cf. |6|. We shall use an exponentially fitted scheme, which in the case of uniform mesh reduces to the well known schemes of |1|. For such a scheme we shall prove the convergence uniform in ε.

2. PROPERTIES OF THE EXACT SOLUTION

A special technique was used in |2| to obtain some differentiability properties of the solution to the boundary value problem with a small parameter, containing the first derivative. We shall here use a similar technique for our problem.

THEOREM 1. *For the solution* u_ε *of* (1) *we have*

$$u_\varepsilon(x) = p_\varepsilon v_\varepsilon(x) + z_{0,\varepsilon}(x), \quad 0 \le x \le 1/2 , \tag{3}$$

$$u_\varepsilon(x) = q_\varepsilon w_\varepsilon(x) + z_{1,\varepsilon}(x), \quad 1/2 \le x \le 1 , \tag{4}$$

where

$$v_\varepsilon(x) = \exp\left(-\frac{b(0)}{\varepsilon}\, x\right) ,$$

$$w_\varepsilon(x) = \exp\left(-\frac{b(1)}{\varepsilon}\, (1-x)\right) ,$$

$$|p_\varepsilon|, |q_\varepsilon| \le M \qquad \textit{and}$$

$$|z^{(i)}_{k,\varepsilon}(x)| \le M(1+\varepsilon^{1-i}), \quad k=0,1 , \quad i=0,1,2,3 . \tag{5}$$

P r o o f. Let $x \in [0,1]$. Define

$$y_\varepsilon(x) = u_\varepsilon(x) - p_\varepsilon v_\varepsilon(x) - q_\varepsilon w_\varepsilon(x)$$

and take p_ε and q_ε so that $y'_\varepsilon(0) = 0$, $y'_\varepsilon(1) = 0$. It is easily obtained that such p_ε, q_ε exist and $|p_\varepsilon|, |q_\varepsilon| \le M$. Hence, we have

$$|y_\varepsilon(x)| \le M.$$

Then we get

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$$L_\varepsilon y_\varepsilon'(x) = g_\varepsilon'(x) - (b^2(x))' y_\varepsilon'(x) =: G_\varepsilon(x) ,$$

where

$$g_\varepsilon(x) = f(x) - p_\varepsilon L_\varepsilon v_\varepsilon(x) - q_\varepsilon L_\varepsilon w_\varepsilon(x) .$$

Now it can be easily proved that, cf. |1|,

$$|g_\varepsilon'(x)| \leq M,$$

which implies $|G_\varepsilon(x)| \leq M$. Hence, $y_\varepsilon'(x)$ is a solution to the problem

$$L_\varepsilon y_\varepsilon'(x) = G_\varepsilon(x) , \quad x \in [0,1],$$

$$y_\varepsilon'(0) = y_\varepsilon'(1) = 0 ,$$

and because of (2), it follows:

$$|y_\varepsilon^{(i)}(x)| \leq M\varepsilon^{1-i}, \quad i=1,2,3, \quad x \in [0,1].$$

Now we can take

$$z_{0,\varepsilon}(x) = y_\varepsilon(x) + p_\varepsilon w_\varepsilon(x), \quad x \in [0,1/2] ,$$

and since

$$|w_\varepsilon(x)| \leq M, \quad x \in [0,1/2] ,$$

we obtain (3) and (5) for k=0. Similarly we take

$$z_{1,\varepsilon}(x) = y_\varepsilon(x) + q_\varepsilon w_\varepsilon(x), \quad x \in [1/2,1] ,$$

to complete the proof.

A similar result was obtained in |1|. For similar estimates of the derivatives of u_ε cf. |3|.

3. THE SCHEME

We use the discretisation mesh

$$I_h = \{x_i \mid x_0 = 0, \; x_i = x_{i-1} + h_i, \quad i=1,2,\ldots,n\} ,$$

where $n \in \mathbb{N}$, $h_i > 0$, $i=1,2,\ldots,n$, $h_1+h_2+\ldots+h_n = 1$. For simplicity we take $n = 2m$, $m \in \mathbb{N}\setminus\{1\}$, and $x_m = 1/2$. Let $h_i \leq h_{i+1}$, $i = 1, 2,\ldots,m-1$, and $h_i \geq h_{i+1}$, $i=m+1,\ldots,n$, which is a natural assumption bacause of boundary layers. Let $h := \max\{h_m, h_{m+1}\} \leq 1/4$.

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Instead of 1/4 we can take any fixed constant in $(0,1/2)$, so this inequality is not a real restriction.

On the mesh I_h we form the discretisation of the problem (1):

$$u_0 = U_0 ,$$

$$(6)\quad L_\varepsilon^h u_i := -\varepsilon^2 \sigma_i D_h u_i + b^2(x_i) u_i = f(x_i),\ i=1,2,\dots,n-1 ,$$

$$u_n = U_1 ,$$

where

$$D_h u_i = \frac{2}{(h_{i+1}+h_i)h_i h_{i+1}} \left(h_{i+1}u_{i-1} - (h_i+h_{i+1})u_i + h_i u_{i+1}\right) ,$$

$$i=1,2,\dots,n-1 ,$$

and

$$\sigma_i = \begin{cases} \sigma_i^0(\rho_0) := \frac{1}{2}\rho_0^2 \dfrac{h_i h_{i+1}(h_i+h_{i+1})}{h_{i+1}(e^{\rho_0 h_{i-1}}) + h_i(e^{-\rho_0 h_{i+1}} - 1)} \\ \qquad i=1,2,\dots,m-1 \\ 1, \qquad i=m \\ \sigma_i^1(\rho_n) := \frac{1}{2}\rho_n^2 \dfrac{h_i h_{i+1}(h_i+h_{i+1})}{h_{i+1}(e^{-\rho_n h_i} - 1) + h_i(e^{\rho_n h_{i+1}} - 1)} \\ \qquad i=m+1,\dots,n-1 \end{cases}$$

$$\rho_i = b(x_i)/\varepsilon ,\quad i=0,1,\dots,n.$$

The solution of (6) is $u_h = [u_0,u_1,\dots,u_n]^T \in \mathbb{R}^{n+1}$ (here we omit the subscript ε). Let u_ε^h denote the restriction of u_ε to the mesh I_h : $u_\varepsilon^h = [u_\varepsilon(x_0), u_0(x_1),\dots,u_\varepsilon(x_n)]^T \in \mathbb{R}^{n+1}$.

We use the maximum norm $\|v\| = \max\limits_{0\le i\le n} |v_i|$ for any vector $v = [v_0,v_1,\dots,v_n]^T \in \mathbb{R}^{n+1}$ and $\|A\| = \max\limits_{0\le i\le n} \sum\limits_{j=0}^{n} |a_{ij}|$ for any matrix $A = [a_{ij}] \in \mathbb{R}^{n+1,n+1}$.

THEOREM 2. *It is valid that*

$$\|u_\varepsilon^h - u_h\| \le M h .$$

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P r o o f. The consistency error of the operator L_ε^h is by definition:

$$r_i := L_\varepsilon^h u_\varepsilon(x_i) - (L_\varepsilon u_\varepsilon)_{x=x_i} = -\varepsilon^2(\sigma_i D_h u_\varepsilon(x_i) - u''_\varepsilon(x_i)) ,$$

$$i = 1,2,\ldots,n-1 .$$

Let $r_h = [0, r_1, r_2, \ldots, r_{n-1}, 0]^T \in \mathbb{R}^{n+1}$ and $f_h = [U_o, f(x_1), f(x_2), \ldots, f(x_{n-1}), U_1]^T \in \mathbb{R}^{n+1}$. Then (6) can be written in the form

$$A_\varepsilon^h u_h = f_h ,$$

where $A_\varepsilon^h \in \mathbb{R}^{n+1,n+1}$ is tridiagonal matrix. Since $\sigma_i > 0$, $i=1,2,\ldots,n-1$, (see Lemma 1. below), A_ε^h is an L-matrix, inverse monotone and (cf. eg. |4|,|5|):

$$\| (A_\varepsilon^h)^{-1} \| \leq \max(1, \beta^{-2}) .$$

Now we have

$$A_\varepsilon^h (u_\varepsilon^h - u_h) = f_h + r_h - r_h = r_h$$

and

$$\| u^h - u_h \| \leq \| (A_\varepsilon^h)^{-1} \| \, \| r_h \| \leq M \| r_h \| .$$

To complete the proof it is sufficiant to show that

$$\| r_h \| \leq M h .$$

or

$$|r_i| \leq M h , \quad i=1,2,\ldots,n-1 . \tag{7}$$

First we consider $i=1,2,\ldots,m-1$. The fitting factor $\sigma_i = \sigma_i^o(\rho_o)$ is chosen so that

$$\sigma_i D_h v_\varepsilon(x_i) = v''_\varepsilon(x_i) ,$$

where $v_\varepsilon(x)$ is given Theorem 1. This fact and (3) imply that

$$|r_i| = \varepsilon^2 |\sigma_i D_h z_{o,\varepsilon}(x_i) - z''_{o,\varepsilon}(x_i)| . \tag{8}$$

Since there exists a point $\theta_i \in [x_{i-1}, x_{i+1}]$ such that

$$D_h z_{o,\varepsilon}(x_i) = z''_{o,\varepsilon}(\theta_i) ,$$

from (8) it follows that

$$|r_i| \leq \varepsilon^2 |1-\sigma_i| |z''_{o,\varepsilon}(\theta_i)| + \varepsilon^2 |z''_{o,\varepsilon}(\theta_i) - z''_{o,\varepsilon}(x_i)| \leq$$

$$\leq \varepsilon^2 |1-\sigma_i| K_2 + \varepsilon^2 h_{i+1} K_3 ,$$

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where

$$K_j = \max_{0\le x\le 1/2} |z^{(j)}_{o,\varepsilon}| , \quad j=2,3 .$$

From Theorem 1. we have

$$K_j \le M\,\varepsilon^{1-j} , \quad j=2,3$$

and

$$|r_i| \le M(\varepsilon|1-\sigma_i| + h) .$$

In Lemma 2. we shall prove that

$$\varepsilon|1-\sigma_i| \le M h_{i+1} .$$

Because of that, (7) is valid for i=1,2,...,m-1.

In the case when i=m+1,...,n-1 the proof of (7) is analogous. We use (4) and

$$\sigma^1_i(\rho_n) D_h w_\varepsilon(x_i) = w''_\varepsilon(x_i) .$$

When i=m, we have

$$|r_m| \le \varepsilon^2 h \max_{x_{m-1}\le x\le x_{m+1}} |u^{(3)}_\varepsilon(x)| ,$$

and because of Theorem 1. :

$$|r_m| \le \varepsilon^2 h M(1+\varepsilon^{-2}+\varepsilon^{-3}(\exp(-\frac{b(0)}{\varepsilon}x_{m-1}) + \exp(-\frac{b(1)}{\varepsilon}(1-x_{m+1})))) .$$

Since $x_{m-1} = 1/2 - h_m \ge 1/4$ and $x_{m+1} = 1/2 + h_{m+1} \le 3/4$, we get

$$|r_m| \le M h .$$

Here we used $\varepsilon^{-1}\exp(-\frac{\beta}{4\varepsilon}) \le M$. This completes the proof of Theorem 2.

LEMMA 1. *For* $\rho > 0$ *it is valid that* $\sigma^o_i(\rho) > 0$, i=1,2,...,m-1.

P r o o f. Let i=1,2,...,m-1 and

$$g(\rho) := h_{i+1}(e^{\rho h} - 1) + h_i(e^{-\rho h_{i+1}} - 1) .$$

We have $g(0) = 0$ and $g'(\rho) > 0$ for $\rho > 0$. Now it follows that $g(\rho) > 0$ and $\sigma^o_i(\rho) > 0$, $\rho > 0$.

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LEMMA 2. *For $\rho > 0$ it is valid that*

$$|1 - \sigma_i^o(\rho)| \leq h_{i+1}\rho , \quad i=1,2,\ldots,m-1 . \tag{9}$$

P r o o f. Let $i=1,2,\ldots,m-1$ and let us denote $k = h_i$, $K_i = h_{i+1}$. First we shall prove

$$\sigma_i^o(\rho) \leq 1 + K\rho . \tag{10}$$

This inequality is equivalent with

$$\sum_{j=3}^{\infty} \frac{\rho^j}{j!} K(k+K) R_j \geq 0 , \tag{11}$$

where

$$R_j = j(k+K)^{j-2}K + (k+K)^{j-1} - (j+1)K^{j-1} - \frac{1}{2} j(j-1)kK^{j-2} .$$

Now we have that $R_3 = k^2 + kK \geq 0$ and from $R_j \geq 0$, $j \geq 3$, it follows that

$$\begin{aligned} R_{j+1} &\geq (k+K)\left((j+1)K^{j-1} + \frac{1}{2} j(j-1)kK^{j-2}\right) + (k+K)^{j-1}K - \\ &- (j+2)K^j - \frac{1}{2}(j+1)jkK^{j-1} \geq \\ &\geq \frac{1}{2} jkK^{j-2}((j-1)(k+K) - (j+1)K) + (j+1)kK^{j-1} = \\ &= \frac{1}{2} j(j-1)k^2K^{j-2} + kK^{j-1} \geq 0 . \end{aligned}$$

Hence, (11) is valid and so is (10). Using a similar technique we can get

$$\sigma_i^o(\rho) \geq 1 - \rho K . \tag{12}$$

The inequalities (11) and (12) imply (9) .

4. SOME REMARKS

The difference scheme used in (6) has quasi-constant fitting factors in the sense of [5]. Namely, when the mesh is uniform ($h_i = h$, $i=1,2,\ldots,n$), we get the constant fitting factors from [1]:

$$\sigma_i^o(\rho_o) = (\rho_o h/2)^2 sh^{-2}(\rho_o h/2), \quad \sigma_i^1(\rho_n) = \sigma_i^o(\rho_n) .$$

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Because of the non-uniform mesh we have proved linear convergence uniform in ε. In |1| it is shown that on the uniform mesh the quadratic convergence uniform in ε can be obtained when $u_\varepsilon \in C^4(I)$ and $b(0) = b(1)$.

A similar scheme with quasi-constant fitting factor was obtained in |5|, but our scheme seems to be simpler.

It is possible to construct a scheme with variable fitting factors. We can take (6) with

$$\sigma_i = \begin{cases} \sigma_i^0(\rho_i), & i=1,2,\ldots,m-1 \\ 1, & i=m \\ \sigma_i^1(\rho_i), & i=m+1,\ldots,n-1 \end{cases}$$

Such a scheme was also considered in |1| on the uniform mesh.

5. NUMERICAL EXAMPLE

We shall consider a simple test problem from |7|,|4|:

$$-\varepsilon^2 u'' + u = 1, \quad x \in I, \quad u(0) = u(1).$$

For the solution of this problem we have $u_\varepsilon(1/2+x) = u_\varepsilon(1/2-x)$, $x \in [0,1/2]$ and we shall give numerical results on $[0,1/2]$.

Let m_1 be the number of mesh points x_i in $[0,\varepsilon]$ and let

$$E = \max\{|u_\varepsilon(x_i) - u_i| \,|\, x_i \in (\varepsilon, 1/2]\},$$

$$E_1 = \max\{|u_\varepsilon(x_i) - u_i| \,|\, x_i \in [0,\varepsilon]\},$$

$$P = \max\{|u_\varepsilon(x_i) - u_i|/|u_\varepsilon(x_i)| \,|\, x_i \in (\varepsilon, 1/2]\} \cdot 100,$$

$$P_1 = \max\{|u_\varepsilon(x_i) - u_i|/|u_\varepsilon(x_i)| \,|\, x_i \in [0,\varepsilon]\} \cdot 100.$$

It is convenient to use such a mesh which changes automatically when ε does. This enables us to achieve the same number of mesh points in the boundary layer whatever value ε takes. A special mesh of a such type was given in |3|, where the mesh points are determined as values of function $\lambda_\varepsilon(s)$ at equidistant points (see |4| as well). This mesh implies a convergence uniform in ε for the classical scheme

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$$-\varepsilon^2 D_h u_i + b^2(x_i)u_i = f(x_i).$$

Here we shall use a similar but simpler function $\lambda_\varepsilon(s)$:

$$\lambda_\varepsilon(s) = \begin{cases} \psi_\varepsilon(s), & s \in [0,\alpha] \\ \psi_\varepsilon(\alpha) + \psi_\varepsilon'(\alpha)(s-\alpha), & s \in [\alpha,1/2] \end{cases}$$

where $\psi_\varepsilon(s) = a\varepsilon s/(q-s)$, $s \in [0,q)$. q and a are fixed constants: $q \in (0,1/2)$ and $a > 0$, and $(\alpha, \psi_\varepsilon(\alpha))$ is the point of contact of a tangent line taking the value 1/2 at s=1/2, to the curve $\psi_\varepsilon(s)$. We take an $\varepsilon \leq q$. Then there exists unique $\alpha \in [0,q)$ and it be easily determined from

$$\psi_\varepsilon(\alpha) = 1/2 + \psi_\varepsilon'(\alpha)(\alpha - 1/2),$$

which reduces to a quadratic equation.

We shall take mesh I_h with

$$x_i = \lambda_\varepsilon(i/n), \quad i=0,1,\ldots,m,$$

and test our scheme (6).

TABLE 1. a=1, q=0.4, m=20

ε	m_1	E	E_1	P	P_1
10^{-2}	9	$1.19\cdot10^{-7}$	$2.09\cdot10^{-7}$	$1.19\cdot10^{-5}$	$3.23\cdot10^{-4}$
10^{-4}	9	$5.96\cdot10^{-8}$	$1.64\cdot10^{-7}$	$8.24\cdot10^{-6}$	$2.54\cdot10^{-4}$
10^{-6}	9	$1.79\cdot10^{-7}$	$2.98\cdot10^{-7}$	$2,47\cdot10^{-5}$	$3.12\cdot10^{-4}$
10^{-10}	9	$1.19\cdot10^{-7}$	$2.09\cdot10^{-7}$	$1.65\cdot10^{-5}$	$3.12\cdot10^{-4}$

TABLE 2. a=0.3, q=0.499, m=20

ε	m_1	E	E_1	P	P_1
10^{-2}	16	$7.75\cdot10^{-7}$	$2.04\cdot10^{-6}$	$1.10\cdot10^{-4}$	$1.89\cdot10^{-3}$
10^{-4}	16	$1.07\cdot10^{-6}$	$2.29\cdot10^{-6}$	$1.53\cdot10^{-4}$	$1.99\cdot10^{-3}$
10^{-6}	16	$1.07\cdot10^{-6}$	$2.40\cdot10^{-6}$	$1.53\cdot10^{-4}$	$2.04\cdot10^{-3}$
10^{-10}	16	$1.19\cdot10^{-6}$	$2.35\cdot10^{-6}$	$1.70\cdot10^{-4}$	$2.10\cdot10^{-3}$

With a small number of mesh points we achieve extremely

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good results. The errors change slightly for different values of ε. A high percentage of mesh points in the boundary layer is obtained.

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REZIME

JEDNA EKSPONENCIJALNO FITOVANA ŠEMA NA NERAVNOMERNOJ MREŽI

U radu se posmatra diskretizacija problema (1) na neravnomernoj mreži. Koristi se eksponencijalno fitovana šema (6)

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dokazuje linearna konvergencija uniformna po malom parametru ε. U slučaju kada je mreža ekvidistantna posmatrana šema svodi se na poznatu šemu iz |1| sa konstantnim faktorom fitovanja.

Numerički rezultati dobijeni su na specijalno konstruisanoj mreži, koja je formirana slično kao u |3|.

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SOME PROPERTIES OF SYNTAX-DIRECTED TRANSLATIONS

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ABSTRACT

In the paper are defined the following concepts: syntax-directed translation scheme (SDTS), form of SDTS, rule of SDTS, syntax-directed translation (SDT), input grammar of SDTS, input language of SDTS, output grammar of SDTS, output language of SDTS, bi-unique SDT, conditions of composition of SDTs, conditions of equality of SDTs, identical SDT, inverse SDT, simple SDTS, simple SDT and compound SDT, and are proved 13 theorems about some properties (equivalence, composition, commutation, association and unity) of SDTs.

INTRODUCTION

Syntax-directed translation has been known for quite a long time. One of the first who has advocated its use was Irons |1|, |2|. There exists a great variety of definition models for syntax directed translations. Two of them are:

- the syntax-directed translation scheme |3| and
- the pushdown transducer |4|.

Syntax-directed translation forms the basis for many compiler generators and compiler writing systems |5|, |6|, |7|, |8|, |9|.

THE BASIC DEFINITIONS AND THEOREMS

DEFINITION 1. *A syntax-directed translation scheme (abbreviation used: SDTS) is a 5-tuple* $T = (N, \Sigma, \Delta, R, S)$, *where:*

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(1) N *is a finite set of nonterminal symbols.*

(2) Σ *is a finite set of terminal symbols (input alphabet).*

(3) Δ *is a finite set of output symbols (output alphabet), and* $N \cap (\Sigma \cup \Delta) = \emptyset$

(4) R *is a finite set of rules of the form:*

$A \to \alpha, \beta, \pi$

where:

$A \in N$,

$\alpha \in (N \cup \Sigma)^*$,

$\beta \in (N \cup \Delta)^*$,

strings α *and* β *contain the same number of nonterminal symbols* m, $\emptyset \leq m \leq \min(|\alpha|, |\beta|)$,

(a) *if* $m \geq 1$ π *is a permutation of indices of the nonterminal symbols of strings* α *and* β, *so that the* i*-th nonterminal symbol of string* α *is equal to the* $\pi(i)$*-th nonterminal symbol of string* β, $1 \leq i, \pi(i) \leq m$. b) *if* $m = \emptyset$ π *is* $[\,]$ *and usually is omitted.*

(5) S *is the start symbol,*

$S \in N$.

DEFINITION 2. *Let* $T = (N, \Sigma, \Delta, R, S)$ *be a SDTS. A form of the SDTS* T *is a triple* $F = (\alpha, \beta, \pi)$, *where* α, β *and* π *satisfy condition (4) of definition 1.*

DEFINITION 3. *Let*

$T = (N, \Sigma, \Delta, R, S)$ *be a SDTS,* T

$F_1 = (\alpha_1, \beta_1, \pi_1)$ *and*

$F_2 = (\alpha_2, \beta_2, \pi_2)$ *be the forms of SDTS* T ,

A *be* i*-th nonterminal symbol of string* α_1 *and* $\pi_1(i)$*-th nonterminal symbol of string* β_1 *and*

$A \to \alpha, \beta, \pi$ *be a rule of the SDTS* T .

Let the string α and β contain m nonterminal symbols

A binary relation of direct derivation of the forms $\Rightarrow_T$ is defined on the set $(N \cup \Sigma)^* \times (N \cup \Delta)^* \times \Pi$, where Π is the set of all the permutations of indices of nonterminal symbols, in the following way:

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- the string α_2 is derived directly by the substitution of i-th nonterminal symbol of the string α_1 with the string α,

- the string β_2 is derived directly by the substitution of $\pi_1(i)$-th nonterminal symbol of the string β_1 with the string β,

- the permutation π_2 is obtained by the permutation π_1 and π in the following way:

(a) if $m \geq 1$ then

(1) for every $j < i$

if $\pi_1(j) < \pi_1(i)$ then $\pi_2(j) = \pi_1(j)$

if $\pi_1(j) > \pi_1(i)$ then $\pi_2(j) = \pi_1(j) + m - 1$;

(2) for every $j > i$

if $\pi_1(j) < \pi_1(i)$ then $\pi_2(j + m - 1) = \pi_1(j)$,

if $\pi_1(j) > \pi_1(i)$ then $\pi_2(j + m - 1) = \pi_1(j) + m - 1$;

(3) $\pi_2(i + d - 1) = \pi_1(i) + \pi(d) - 1$

for every d, $1 \leq d \leq m$.

(b) if $m = \emptyset$ then

$\pi = [\ ]$ but π_2 is obtained by the permutation π_1, applying the rules (1) and (2).

THEOREM 1. *The relation of direct derivation is:*

- *reflexive,*
- *transitive and*
- *nonsymmetric.*

P r o o f. By definition, a binary relation R on a set S is reflexive if and only if (abbreviation used: iff) for every $s \in S$ valid sRs.

From here the binary relation $\Rightarrow_T$ on the set $(N \cup \Sigma)^* \times (N \cup \Delta)^* \times \Pi$ is reflexive iff for every form $(\alpha,\beta,\pi) \in (N \cup \Sigma)^* \times (N \cup \Delta)^* \times \Pi$ valid $(\alpha,\beta,\pi) \Rightarrow_T (\alpha,\beta,\pi)$.

Let us suppose the opposite, i.e. that there exists the form $(a,b,p) \in (N \cup \Sigma)^* \times (N \cup \Delta)^* \times \Pi$ for which the binary

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relation $\Longrightarrow_T$ is not valid. As follows from rule $A \to A,A,[1]$ applied on the form (a,b,p) gives the form (a',b',p') thus the minimum one of the following three inequalities:

$$a \neq a'$$
$$b \neq b'$$
$$p \neq p' \; ; \quad p' = p \cdot [1]$$

must be valid, but it is impossible.

In a similar way we can prove two other properties.

The transitive closure of the relation $\Longrightarrow_T$ is denoted by $\overset{+}{\Longrightarrow}_T$.

The reflexive and transitive closure of the relation $\Longrightarrow_T$ is denoted by $\overset{*}{\Longrightarrow}_T$.

The k-th product of the relation $\Longrightarrow_T$ is denoted by $\overset{k}{\Longrightarrow}_T$.

In the cases when a mistake can not be made, it is possible to omit the symbol T of SDTS.

DEFINITION 4. *Syntax-directed translation (abbreviation used: SDT) is the translation defined by SDTS.*

Let $T = (N,\Sigma,\Delta,R,S)$ be a SDTS.

SDT $\tau(T) = \{(x,y) \mid (S,S) \overset{*}{\Longrightarrow}_T (x,y);\ x \in \Sigma^*,\ y \in \Delta^*\}$.

DEFINITION 5. *Let* $T = (N,\Sigma,\Delta,R,S)$ *be a SDTS.*

The grammar $G_i = (N,\Sigma,P_i,S)$, *where* $P_i = \{A \to \alpha \mid A \to \alpha,\beta,\pi,\ \pi, \in R\}$, *is the input grammar of SDTS* T. *The language* $L_i = L(G_i)$ *defined by the grammar* G_i, *that is,* $L_i = \{x \mid S \overset{*}{\Longrightarrow} x, x \in \Sigma^*\}$ *is the input language of SDTS* T, *that is, a language from which is translated.*

The grammar $G_o = (N,\Delta,P_o,S)$, *where* $P_o = \{A \to \beta \mid A \to \alpha,\beta,\pi \in R\}$, *is the output grammar of SDTS* T. *The language* $L_o = L(G_o)$ *defined by the grammar* G_o, *that is,* $L_o = \{y \mid S \overset{*}{\Longrightarrow} y,\ y \in \Delta^*\}$ *is the output language of SDTS* T, *namely a language into which is translated.*

The derivation rules of the grammar G_i are in the form:

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$$A \to \alpha; \quad A \in N, \quad \alpha \in (N \cup \Sigma)^* ,$$

so the grammar G_i is contex-free. Consequently, the language L_i is contex-free.

The derivation rules of the grammar G_o are in the form:

$$A \to \beta; \quad A \in N, \quad \beta \in (N \cup \Delta)^* ,$$

so the grammar G_o is contex-free. Consequently, the language L_o is the contex-free.

THEOREM 2. *The domain of defining* $X \subseteq \Sigma^*$ *and the set of value* $Y \subseteq \Delta^*$ *of the SDT* $\tau(T)$ *are equal, respectively, to the input and the output language of the SDTS* T .

P r o o f. Let $T = (N, \Sigma, \Delta, R, S)$ be an SDTS. The SDTD T defines the SDT $\tau(T)$:

$$\tau(T) = \{(x,y) \mid (S,S) \overset{*}{\underset{T}{\Longrightarrow}} (x,y), \quad x \in \Sigma^* , \quad y \in \Delta^*\} .$$

The domain of defining the SDT $\tau(T)$ is the set X:

$$X = \{x \mid S \overset{*}{\underset{G_i}{\Longrightarrow}} x, \quad x \in \Sigma^*\} .$$

The set of values of the SDT $\tau(T)$ is the set Y:

$$Y = \{y \mid S \overset{*}{\underset{G_o}{\Longrightarrow}} y, \quad y \in \Delta^*\} .$$

It follows directly that:

$$X = L_i \quad \text{and}$$
$$Y = L_o$$

as we wanted to prove.

COROLLARY. *Let* X *and* Y *be some languages. If* $\min(\text{tip}(X), \text{tip}(Y)) \geq 2$ *then, in a general case, there exist a SDT* $\tau_1 = \tau(T_1)$ *and a SDT* $\tau_2 = \tau(T_2)$, *so it is valid:*

$$\tau_1 : X \to Y$$
$$\tau_2 : Y \to X ,$$

besides, there exist τ_1 *and* τ_2 *but only in special cases.*

Speaking differently the theory of SDT is valid in a general case for contex-free and regular languages.

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IMPORTANT PROPERTIES OF SDT

THEOREM 3. *Let* $\tau_1 = \tau(T_1)$ *and* $\tau_2 = \tau(T_2)$ *be the SDT, respectively defined by the SDTS* T_1 *and* T_2:

$$T_1 = (N_1, \Sigma, \Delta, R_1, S_1)$$

$$T_2 = (N_2, \Sigma, \Delta, R_2, S_2)$$

If:

$$L_i^{(1)} = L_i^{(2)} = L_i \subseteq \Sigma^* \quad \textit{and}$$

$$\tau_1(x) = \tau_2(x)$$

for every word x *from the domain* L_i, *then it is:*

$$L_o^{(1)} = L_o^{(2)} = L_o \subseteq \Delta^* .$$

P r o o f. By definition, we have:

$$\tau_1 = \tau(T_1) = \{(x,y) \mid (S_1, S_2) \overset{*}{\underset{T_1}{\Longrightarrow}} (x,y),\ x \in L_i,\ y \in L_o^{(1)}\}$$

and

$$\tau_2 = \tau(T_2) = \{(x,y) \mid (S_2, S_2) \overset{*}{\underset{T_2}{\Longrightarrow}} (x,z),\ x \in L_i,\ z \in L_o^{(2)}\} .$$

Let us have:

$$\tau_1(x) = y \in L_o^{(1)} \subseteq \Delta^*$$

$$\tau_2(x) = z \in L_o^{(2)} \subseteq \Delta^*$$

Since:

$$\tau_1(x) = \tau_2(x)$$

it follows:

$$y = z.$$

As, it is valid for every word $x \in L_i$, and it follows immediately:

$$L_o^{(1)} = L_o^{(2)} .$$

THEOREM 4. *In a general case, the problem of equivalence of two SDT is undecidable.*

P r o o f. Let $\tau_1 = \tau(T_1)$ and $\tau_2 = \tau(T_2)$ be two SDT, respectively, defined by the SDTS T_1 and T_2:

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$$T_1 = (N_1, \Sigma, \Delta, R_1, S_1)$$

and

$$T_2 = (N_2, \Sigma, \Delta, R_2, S_2)$$

Further, let:

$$L_i^{(1)},\ L_o^{(1)},\ L_i^{(2)} \text{ and } L_o^{(2)}$$

be input and output language, that are the domains and the ranges of SDT τ_1 and τ_2.

The mentioned languages are contex-free. The problem of equivalence of two SDT can be reduced to the problem of equivalence:

$$L_i^{(1)} = L_i^{(2)}$$

and

$$L_o^{(1)} = L_o^{(2)}$$

of two pairs of contex-free languages, which are undecidable in a general case, as it is well-known.

DEFINITION 6. *Let τ be a SDT. τ is a bi-unique translation iff the following relation of equivalence is valid:*

$$\forall(x_1, x_2)(x_1 \cdot L_i,\ x_2 \cdot L_i)\tau(x_1) = \tau(x_2) \Longleftrightarrow x_1 = x_2 .$$

THEOREM 5. *Let $\tau = \tau(T)$ be a SDT defined by the SDTS T:*

$$T = (N, \Sigma, \Delta, R, S)$$

τ is a bi-unique translation iff for every rules of the SDTS T of the form:

$$A \to \alpha_1, \beta_1, \pi_1$$

and

$$A \to \alpha_2, \beta_2, \pi_2$$

is valid in the following equivalence relation:

$$\alpha_1 = \alpha_2 \Longleftrightarrow \beta_1 = \beta_2 .$$

π_1 can be equal to π_2, but it is not necessary.

P r o o f. Let τ be a bi-unigue translation. Let:

$$(S, S, [1]) = F_o \Longrightarrow F_1 \Longrightarrow \ldots \Longrightarrow F_n = (x_1, \tau(x_1), [\ ])$$

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also be the derivation and translation of the word x_1.

Let us suppose, that by using the rule $A \to \alpha_1, \beta_1, \pi_1$ the form F_{i+1} is directly derived from the form F_i, namely:

$$F_i = (a_i, b_i, p_i) \Rightarrow (a_{i+1}, b_{i+1}, p_{i+1}) = F_{i+1}$$

and by using the rule $A \to \alpha_2, \beta_2, \pi_2$ the form F'_{i+1} is directly derived from the form F_i, namely:

$$F_i = (a_i, b_i, p_i) \Rightarrow (a'_{i+1}, b'_{i+1}, p'_{i+1}) = F'_{i+1}.$$

If $\alpha_1 = \alpha_2$ and $\beta_1 \neq \beta_2$, it follows that $b_i \neq b_{i+1}$, so as the result of the derivation and translation, $(x_1, \tau(x_1)', [\,])$ is obtained, where $\tau(x_1) \neq \tau(x_1)'$. And that is impossible, since τ is a bi-unique translation. So it results that if $\alpha_1 = \alpha_2$ then $\beta_1 = \beta_2$.

Analogously, for the assumption $\alpha_1 \neq \alpha_2$, $\beta_1 \neq \beta_2$ we obtain the contradiction too, hence we exclude this assumption.

DEFINITION 7. *Let* $\tau_i = \tau(T_i)$, $i=1,\ldots,n$ *be an SDT defined by the SDTS* $T_i = (N_i, \Sigma_i, \Delta_i, R_i, S_i)$, $i=1,\ldots,n$.

The conditions:

- $N_i = N_{i+1} = N$, $i=1,\ldots,n-1$
- $\Delta_i = \Sigma_{i+1}$, $i=1,\ldots,n-1$
- *for every rule* $A \to \beta_{i-1}, \beta_i, \pi_i \in R_i$ *there exists the rule* $A \to \beta_i, \beta_{i+1}, \pi_{i+1} \in R_{i+1}$, $i=2,\ldots,n-1$
- $S_i = S_{i+1} = S$, $i=1,\ldots,n-1$

are called the conditions of composition SDT τ_i, $i=1,\ldots,n$.

THEOREM 6. *The composition* $\tau = \tau_1 \cdot \ldots \cdot \tau_n$ *of the SDT* τ_i, $i=1,\ldots,n$, *which satisfies the conditions of the composition is SDT.*

P r o o f. First of all, let us prove that the composition $\tau = \tau_1 \cdot \tau_2$ of SDT τ_1 and τ_2 is SDT satisfying the conditions of the composition.

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In fact, by using the definition of SDTS, SDT and the conditions of the composition of SDT, we obtain:

$$T_1 = (N, \Sigma_1, \Delta, R_1, S)$$

$$T_2 = (N, \Delta, \Delta_2, R_2, S)$$

$$\tau_1 = \tau(T_1) = \{(x,y) \mid (S,S) \overset{*}{\underset{T_1}{\Longrightarrow}} (x,y),\ x \in \Sigma_1^*,\ y \in \Delta^*\}$$

$$\tau_2 = \tau(T_2) = \{(y,z) \mid (S,S) \overset{*}{\underset{T_2}{\Longrightarrow}} (y,z),\ y \in \Delta^*,\ z \in \Delta_2^*\}\ .$$

Let be:

$$T = (N, \Sigma_1, \Delta_2, R, S),$$

where

$$R = \{A \to \alpha,\gamma,\pi \mid A \to \alpha,\beta,\pi_1 \in R_1,\ A \to \beta,\gamma,\pi_2 \in R_2,\ \pi = \pi_1 \cdot \pi_2\}$$

and

$$\tau = \tau(T) = \{(x,z) \mid (S,S) \overset{*}{\underset{T}{\Longrightarrow}} (x,z),\ x \in \Sigma_1^*,\ z \in \Delta_2^*\}\ .$$

T is the SDTS of τ. The SDT τ is the composition of SDT τ_1 and τ_2.

In fact, let:

$$x \in \Sigma_1^*$$

$$y = \tau_1(x) \in \Delta_1^* \quad \text{and}$$

$$z = \tau_2(y) \in \Delta_2^*\ .$$

$(x,y) \in \tau_1$ is obtained by $y = \tau_1(x)$, that means that there exists the derivation:

$$(S,S) = F_o \Longrightarrow F_1 \Longrightarrow \ldots \Longrightarrow F_i = (\alpha_i, \beta_i, p_i) \Longrightarrow \ldots \Longrightarrow F_n = (x,y).$$

Analogously, $(y,z) \in \tau_2$ is obtained by $z = \tau_2(y)$, and that also means that there exists the derivation:

$$(S,S) = G_o = G_1 \Longrightarrow \ldots \Longrightarrow G_i = (\beta_i, \gamma_i, q_i) \Longrightarrow \ldots \Longrightarrow G_n = (y,z).$$

So, it follows from here that exists the following derivation:

$$(S,S) = H_o \Longrightarrow H_1 \Longrightarrow \ldots \Longrightarrow H_i = (\alpha_i, \gamma_i, r_i) \Rightarrow \ldots \Longrightarrow H_n = (x,z)$$

which signify that $(x,z) \in \tau = \tau_1 \cdot \tau_2$, and that $z = \tau(x) = \tau_2(\tau_1(x)) =$

$= (\tau_1 \cdot \tau_2)(x)$ and τ is SDT.

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According to the mathematical induction there follows directly the proof of the theorem for n SDT.

DEFINITION 8. *Let* $\tau_i = \tau(T_i)$, $i=1,\ldots,n$ *be an SDT defined by the SDTS* $T_i = (N, \Sigma_i, \Delta_i, R_i, S)$, $i=1,\ldots,n$.

The conditions:

$$\Sigma_1 = \ldots \Sigma_n = \Delta_1 = \ldots = \Delta_n$$

are called the conditions of equality of SDT τ_i, $i=1,\ldots,n$.

The SDTSs which satisfy the conditions of equality are of the form:

$$T_i = (N, \Sigma, \Delta, R_i, S), \quad i=1,\ldots,n.$$

THEOREM 7. *Let* $\tau_1 = \tau(T_1)$ *and* $\tau_2 = \tau(T_2)$ *be the SDTs which satisfy the conditions of composition and equality.*

In a general case the law of commutation is not valied for SDTs, i.e. $\tau_1 \cdot \tau_2 \neq \tau_2 \cdot \tau_1$.

P r o o f. Let, us suppose the opposite and then prove the claim of the theorem by guoting the contra-example.

Let:

$$T_1 = (\{<bb>,<bc>\},\{0,1\},\{0,1\},R_1,<bb>)$$

where

$$R_1 = \{<bb> \to <bc>,<bc>$$
$$<bb> \to <bb><bc>,<bb><bc>$$
$$<bc> \to 0,0$$
$$<bc> \to 1,0\}$$

be an SDTS.

$$\tau_1 = \tau(T_1) = \{(x,y) \mid x \in \{0,1\}^+,\ y \in \{0\}^+,\ |x| = |y|\}$$

is the SDT defined by the SDTS T_1.

Also, let:

$$T_2 = (\{<bb>,<bc>\},\{0,1\},\{0,1\},R_2,<bb>)$$

where:

$$R_2 = \{<bb> \to <bc>,\ <bc>$$
$$<bb> \to <bb><bc>,<bb><bc>$$
$$<bc> \to 0,1$$
$$<bc> \to 1,1\}$$

be another SDTS.

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$$\tau_2 = \tau(T_2) = \{(x,y) \mid x \in \{0,1\}^+,\ y \in \{1\}^+,\ |x| = |y|\}$$

is the SDT defined by the SDTS T_2.

For every x, $x \in \{0,1\}^+$

$$(\tau_1 \cdot \tau_2)(x) = \tau_2(\tau_1(x)) = \tau_2(y) = z$$

$y \in \{0\}^+$, $|x| = |y|$ and

$z \in \{1\}^+$, $|y| = |z|$.

$$(\tau_2 \cdot \tau_1)(x) = \tau_1(\tau_2(x)) = \tau_1(z) = y$$

$z \in \{1\}^+$, $|x| = |z|$ and

$y \in \{0\}^+$, $|z| = |y|$.

Since $z \neq y$, it follows that $\tau_1 \cdot \tau_2 \neq \tau_2 \cdot \tau_1$.

THEOREM 8. *Let $\tau_1 = \tau(T_1)$, $\tau_2 = \tau(T_2)$ and $\tau_3 = \tau(T_3)$ be the SDTs which satisfy the conditions of the composition.*

In general case the law of association is validated for the SDT, i.e. $(\tau_1 \cdot \tau_2) \cdot \tau_3 = \tau_1 \cdot (\tau_2 \cdot \tau_3)$.

P r o o f. Let:

$$T_1 = (N, \Sigma_1, \Delta_1, R_1, S) ,$$

$$T_2 = (N, \Delta_1, \Delta_2, R_2, S) \text{ and}$$

$$T_3 = (N, \Delta_2, \Delta_3, R_3, S)$$

be the corresponding SDTSs.

For every derivation and translation rule.

$$A \to \alpha, \beta, \pi_1 \in R$$

there are derivation and translation rules:

$$A \to \beta, \gamma, \pi_2 \in R_2 \quad \text{and}$$

$$A \to \gamma, \delta, \pi_3 \in R_3 .$$

Furthermore, let:

(1) $\tau_1(x) = y$

(2) $\tau_2(y) = z$

(3) $\tau_3(z) = u$.

From (1) it follows that there is the derivation:

(1´) $(S,S) = F_o \Rightarrow \ldots \Rightarrow (\alpha_i, \beta_i) = F_i \Rightarrow \ldots \Rightarrow (x,y)$.

Analogously, from (2) and (3) it follows:

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(2´) $(S,S) = G_o \Rightarrow \ldots \Rightarrow (\beta_i,\gamma_i) = G_i \Rightarrow \ldots \Rightarrow (y,z)$

and

(3´) $(S,S) = H_o \Rightarrow \ldots \Rightarrow (\gamma_i,\delta_i) = H_i \Rightarrow \ldots \Rightarrow (z,u)$.

From (1´) and (2´) it follows:

(12) $(S,S) \Rightarrow \ldots \Rightarrow (\alpha_i,\gamma_i) \Rightarrow \ldots \Rightarrow (x,z)$

and from (2´) and (3´) it follows:

(23) $(S,S) \Rightarrow \ldots \Rightarrow (\beta_i,\delta_i) \Rightarrow \ldots \Rightarrow (y,u)$.

Further that means that

$$(\tau_1 \cdot \tau_2)(x) = z$$

and

$$(\tau_2 \cdot \tau_3)(y) = u$$

so:

$$((\tau_1 \cdot \tau_2) \cdot \tau_3)(x) = \tau_3((\tau_1 \cdot \tau_2)(x)) = \tau_3(z) = u$$

$$(\tau_1 \cdot (\tau_2 \cdot \tau_3))(x) = (\tau_2 \cdot \tau_3)(\tau_1(x)) = (\tau_2 \cdot \tau_3)(y) = u$$

and finally:

$$((\tau_1 \cdot \tau_2) \cdot \tau_3)(x) = (\tau_1 \cdot (\tau_2 \cdot \tau_3))(x)$$

and this is the claim of the theorem.

DEFINITION 9. *SDT* τ_I *which is defined by the SDTS* T_I *:*

$$T_I = (N,\Sigma,\Sigma,R_I,S)$$

where:

$$R_I = \{A \to \alpha, \alpha, \pi_I\},$$

$$A \in N,$$

$$\alpha \in (N \cup \Sigma)^*$$

$\pi_I = [1,\ldots,m]$, *where* m *is the number of the nonterminal symbols of the string* α *is an identical translation.*

DEFINITION 10. *Let* τ *and* θ *be the SDTs. If:*

$$\tau \cdot \theta = \tau_I \quad \text{and} \quad \theta \cdot \tau = \tau_I$$

then the translation θ *is the inverse translation of the translation* τ .

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The inverse translation is denoted by the same letter and upper index -1, i.e. $\theta = \tau^{-1}$.

$\tau \cdot \tau^{-1}$ is the identical translation of the language Σ^*.

$\tau^{-1} \cdot \tau$ is the identical translation of the language Δ^*.

THEOREM 9. *Let* $T = (N, \Sigma, \Delta, R, S)$ *be an SDTS of a bi-unique translation*

$$\tau = \tau(T)$$

$$\tau = \{(x,y) \mid (S,S) \overset{*}{\Rightarrow}_T (x,y),\ x \in \Sigma^*,\ y \in \Delta^*\} .$$

$T^{-1} = (N, \Delta, \Sigma, R^{-1}, S)$ *is the SDTS of the SDT*

$$\tau^{-1} = \tau(T^{-1})$$

$$\tau^{-1} = \{(y,x) \mid (S,S) \overset{*}{\Rightarrow}_{T^{-1}} (y,x),\ y \in \Delta^*,\ x \in \Sigma^*\}$$

where:

$$R^{-1} = \{A \to \beta, \alpha, \Pi^{-1} \mid A \to \alpha, \beta, \Pi \in R,\ \Pi \cdot \Pi^{-1} = \Pi_I\} .$$

P r o o f. The translation τ^{-1} is the inverse translation of the translation τ, if the following cases are satisfied:

$$\tau \cdot \tau^{-1} = \tau_I \quad \text{and}$$

$$\tau^{-1} \cdot \tau = \tau_I .$$

Since:

$$T = (N, \Sigma, \Delta, R, S) \quad \text{and}$$

$$\tau(T) = \{(x,y) \mid (S,S) \overset{*}{\Rightarrow}_T (x,y),\ x \in \Sigma^*,\ y \in \Delta^*\}$$

and

$$T^{-1} = (N, \Delta, \Sigma, R^{-1}, S) \quad \text{and}$$

$$\tau(T^{-1}) = \{(y,x) \mid (S,S) \overset{*}{\Rightarrow}_{T^{-1}} (y,x),\ y \in \Delta^*,\ x \in \Sigma^*\}$$

it follows that:

$$T \cdot T^{-1} = (N, \Sigma, \Sigma, R \cdot R^{-1}, S) \quad \text{and}$$

$$\tau(T) \cdot \tau(T^{-1}) = \{(x,x) \mid (S,S) \overset{*}{\Rightarrow}_{T \cdot T^{-1}} (x,x),\ x \in \Sigma^*\} = \tau(T \cdot T^{-1})$$

and

$$T^{-1} \cdot T = (N, \Delta, \Delta, R^{-1} \cdot R, S) \quad \text{and}$$

$$\tau(T^{-1}) \cdot \tau(T) = \{(y,y) \mid (S,S) \overset{*}{\Rightarrow}_{T^{-1} \cdot T} (y,y),\ y \in \Delta^*\} = \tau(T^{-1} \cdot T).$$

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Since:

$$R = \{A \to \alpha,\beta,\Pi \mid A \in N,\ \alpha \in (N \cup \Sigma)^*,\ \beta \in (N \cup \Delta)^*\} \quad \text{and}$$

$$R^{-1} = \{A \to \beta,\alpha,\Pi^{-1} \mid A \to \alpha,\beta,\Pi \in R,\ \Pi \cdot \Pi^{-1} = \Pi_I\}$$

it follows that:

$$R \cdot R^{-1} = \{A \to \alpha,\alpha,\Pi_I\} = R_I$$

and

$$R^{-1} \cdot R = \{A \to \beta,\beta,\Pi_I\} = R_I$$

then:

$$T \cdot T^{-1} = (N,\Sigma,\Sigma,R_I,S) = T_I$$

and

$$T^{-1} \cdot T = (N,\Delta,\Delta,R_I,S) = T_I$$

then

$$\tau \cdot \tau^{-1} = \tau(T) \cdot \tau(T^{-1}) = \tau(T \cdot T^{-1}) = \tau(T_I) = \tau_I$$

and

$$\tau^{-1} \cdot \tau = \tau(T^{-1}) \cdot \tau(T) = \tau(T^{-1} \cdot T) = \tau(T_I) = \tau_I ,$$

and that is the claim of the theorem.

THEOREM 10. *Let τ be a bi-unique SDT. The inverse translation τ^{-1} is unique.*

P r o o f. If we suppose the opposite, i.e. let τ_1 and τ_2 be two mutually different inverse translations of the translation τ.

$\tau_1 \cdot \tau$ and $\tau_2 \cdot \tau$ are identical translations of the language Δ^*, i.e. $\tau_1 \cdot \tau(y) = \tau_I(y) = y$, for every $y \in \Delta^*$

$$\tau_2 \cdot \tau(y) = \tau_I(y) = y, \text{ for every } y \in \Delta^*$$

then $$\tau_1 \cdot \tau(y) = \tau_2 \cdot \tau(y) , \text{ for every } y \in \Delta^*$$

$$\tau(\tau_1(y)) = \tau(\tau_2(y)), \text{ for every } y \in \Delta^* .$$

Since τ is a bi-unique SDT, it follows, that:

$$\tau_1(y) = \tau_2(y), \text{ for every } y \in \Delta^*$$

then $$\tau_1 = \tau_2$$

and that is the opposite to the initial assumption.

It follows, that:

$$\tau_1 = \tau_2 \neq \tau^{-1} .$$

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THEOREM 11. *Let* τ *not be a bi-unique SDT. The SDT* τ^{-1} *is not unique.*

P r o o f. We shall prove the theorem by quotation of a contra-example.

Let:

$$T = (\{<bb>,<bc>\},\{0,1\},\{0,1\},R,<bb>)$$

be an SDTS, where:

$$R = \{<bb> \to <bc>,<bc>$$
$$<bb> \to <bb><bc>,<bb><bc>$$
$$<bc> \to 0,0$$
$$<bc> \to 1,0\} .$$

$$\tau = \tau(T) = \{(x,y) \mid x \in \{0,1\}^+,\ y \in \{0\}^+,\ |x| = |y|\},$$

is the SDT, which is defined by the SDTS T.

$$T^{-1} = (\{<bb>,<bc>\},\{0,1\},\{0,1\},R^{-1},<bb>)$$

is the inverse SDTS, where:

$$R^{-1} = \{<bb> \to <bc>,<bc>$$
$$<bb> \to <bb><bc>,<bb><bc>$$
$$<bc> \to 0,0$$
$$<bc> \to 0,1\} .$$

The following rules:

$$<bc> \to 0,0 \quad \text{and}$$
$$<bc> \to 0,1$$

are the elements of the set R^{-1} and they cause the inuniqueness.

For example:

$$\tau(01) = 00 \text{ , but}$$

$$\tau^{-1}(00) = \begin{cases} 00 \\ 01 \\ 10 \\ 11 \end{cases} .$$

DEFINITION 11. *SDTS* $T = (N, \Sigma, \Delta, R, S)$ *is a simple SDTS if in every rule* $A \to \alpha,\beta$ *of the set* R, *the nonterminal symbols of the strings* α *and* β *follow in the same order.*

DEFINITION 12. *SDT* τ *is simple iff it is defined by a simple SDTS* $T(\tau)$.

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THEOREM 12. *The composition of:*

- *a simple and compound,*
- *a compound and simple*

SDT is a compound SDT.

The composition of a finite number simple SDT is a simple SDT.

The composition of a compound and compound SDT is a compound SDT if the given translations are not mutually inverse. In the opposite case the composition is a simple SDT.

P r o o f. Let $T = (N, \Sigma, \Delta, R, S)$ be an SDTS, and $\tau = \tau(T)$ the corresponding SDT.

If every rule of SDTS T has the form:

$$A \to \alpha, \beta, \Pi_I$$

then SDTS T and SDT τ are simple.

Let:

$$A \to \alpha, \beta, \Pi_1 \text{ and}$$

$$A \to \beta, \gamma, \Pi_2$$

be, in that order, the rules of SDTS T_1 and SDTS T_2.

The composition of the first and second rule is the following rule:

$$A \to \alpha, \gamma, \Pi = \Pi_1 \cdot \Pi_2$$

of the SDTS $T = T_1 \cdot T_2$

$$\Pi = \Pi_I \cdot \Pi = \Pi \cdot \Pi_I, \quad \Pi \neq \Pi_I$$
$$\Pi_I = \Pi_I \cdot \Pi_I$$
$$\Pi_I \neq \Pi = \Pi_1 \cdot \Pi_2, \quad \Pi_1 \neq \Pi_I, \quad \Pi_2 \neq \Pi_I$$
$$\Pi_I = \Pi = \Pi_1 \cdot \Pi_2, \quad \Pi_1 \neq \Pi_I, \quad \Pi_2 \neq \Pi_I, \quad \Pi_1 = \Pi_2^{-1}, \quad \Pi_2 = \Pi_1^{-1}$$

which follows the truth of the claim of the theorem.

THEOREM 13. *Let τ be a simple SDT. The inverse translation of a simple SDT is a simple SDT.*

P r o o f. Let $T = (N, \Sigma, \Delta, R, S)$ be the scheme of a simple SDT $\tau = \tau(T)$. The rules of the SDTS T have the following form:

$$A \to \alpha, \beta, \Pi_I .$$

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The scheme T^{-1} of the inverse SDT $\tau^{-1} = \tau(T^{-1})$ has the following form:

$$T^{-1} = (N, \Delta, \Sigma, R^{-1}, S)$$

where is:

$$R^{-1} = \{A \to \beta, \alpha, \Pi^{-1} \mid A \to \alpha, \beta, \Pi_I \in R,\ \Pi_I \cdot \Pi^{-1} = \Pi_I\}.$$

Since:

$\Pi = \Pi_I \cdot \Pi$ for every permutation Π, it follows, that:

$\Pi_I \cdot \Pi^{-1} = \Pi^{-1} = \Pi_I$.

The rules of the set R^{-1} have the following form:

$$A \to \beta, \alpha, \Pi_I$$

then the scheme T^{-1} and the translation τ^{-1} are simple.

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REZIME

NEKE OSOBINE SINTAKSNO-VODJENIH PREVODJENJA

U radu su definisani sledeći pojmovi: šema SVP (sintaksno-vodjenog prevodjenja), forma, SVP, obostrano-jednoznačno SVP, indentičko SVP, inverzno SVP, prosto SVP, složeno SVP kao i uslovi kompozicije i jednakosti SVP.

Dokazano je više teorema u vezi osobina SVP:

- Oblast definisanosti i skup vrednosti SVP su KS-jezici.
- Problem ekvivalentnosti dva SVP je nerešiv.
- Kompozicija SVP, koja zadovoljavaju uslove kompozicije je SVP.
- Ne važi zakon komutacije za SVP.
- Važi zakon asocijacije za SVP, koja zadovoljavaju uslove kompozicije.
- Inverzno prevodjenje, obostrano-jednoznačnog SVP je jedinstveno.
- Inverzno prevodjenje SVP je u opštem slučaju višeznačno.
- Kompozicija prostog i složenog (odnosno složenog i prostog) SVP je složeno SVP.
- Kompozicija konačno mnogo prostih SVP je prosto SVP.
- Kompozicija složenog i složenog SVP je složeno SVP ako prevodjenja nisu medjusobno inverzna - u suprotnom je prosto SVP.
- Inverzno prevodjenje prostog SVP je prosto SVP.

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SOME PROOFS BY STRUCTURAL INDUCTION ON LISTS

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ABSTRACT

In the paper, the sets Ω and ω are defined in the following way:

a) Ω is the set of the lists of the form:

$[e_1,e_2,\ldots,e_n,\ldots]$, $n \to \infty$

Each element of the list can be:

an atom, a finite list, an element of the set Ω and an element of the set ω .

b)ω is the set of lists which satisfy one of the following conditions:

- they are the elements of the set Ω
- some of its sublist is the element of the set Ω
- they contain an infinite number of sublist.

The functions length and len are defined in the following way:

length [ ] = 0

length (a:x) = 1 + length x

len [ ] = 0

len a = 0; atom a

len (a:x) = 1 + len a + len x .

The main results are:

length x $\to \infty$ $\Leftrightarrow$ x $\in \Omega$ and

len x $\to \infty$ $\Leftrightarrow$ x $\in \omega$.

The functions lenght(len) is used for the examination: the value of the function, whose arguments are the elements of the set $\Omega(\omega)$, is the element of the set $\Omega(\omega)$.

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LISTS

Lists are ordered collection of arbitrary many elements which can be accessed by position. The elements of lists may be of any type including numbers, strings (atoms) or other lists (sublists). Mixed type of elements and duplicating are permitted too.

The finite lists is a list which contains a finite number of elements. The finite list is written in this way:

$[e_1,e_2,\ldots,e_n]$, where n is the number of elements.

The Ω-list is a lists which contains an infinite number of elements. The Ω-list is written in this way:

$[e_1,e_2,\ldots,e_n]$, $n \to \infty$.

The set of all Ω-lists is denoted by Ω ($|2,3,4|$). The ω-list is a list which contains an infinite number of atoms and or an infinite number of sublists.

The set of all ω-lists is denote by ω .

The list which contains no elements(an empty lists) is written in this way:

$[\,]$.

SOME ELEMENTARY FUNCTIONS ON LISTS

Lists may be the arguments and results of functions. The basic construction operator on lists is ":". It makes a list one unit longer by prefixing an element to it. For example:

$1:[2,3]$

has the value $[1,2,3]$.

The "cons" function may be defined as follows:

cons a x = a:x

where:

a is an atom or a list,

x is a list.

The formal parameters need not be simple variables but also the expressions built out of the variables and constants using the ":" operator.

This use of expressions reduces the need for explicit guards, and makes the definitions of functions, by the system of equations, more readable.

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A way for decomposing a list is by using functions that yield the head or the tail of a given list.

The selector functions on the lists are the "hd" and "tl" functions.

The "hd" function yield the first element of the list. It may be defined as follows:

hd(a:x) = a

For example:

hd([1,2,3]) = hd(1:[2,3]) = 1.

The "tl" function yields the list which remains when the head element of the argument list is removed. It may be defined as follows:

tl(a:x) = x

For example:

tl([1,2,3]) = tl(1:[2,3]) = [2,3] .

Notice the differences between the results of the functions: "hd" yeilds an element of the same type as the first element of the list whereas "tl" yields a list.

Both functions have a restricted domain in that they may be only applied to non-empty lists. Thus:

hd [ ] and

tl [ ]

are both undefined.

Let us consider some basic functions.

The "lenght" function yields the number of elements in a list. The "length" function may be defined as follows:

length [ ] = 0 ... length.1

length(a:x)=1+length x length.2

For example:

length([1,2,3]) = 3 .

The length of a list is the number of elements (including duplicates) in the list to which the function is applied.

An (infix) cocnatenation operator can be used to create a list which contains all the elements of its first operand, followed by all the elements of the second operand. The concatenat - on operator "++" may be defined as follows:

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```
[]++y = y                                    ... ++.1
(a:x)++ y = a : (x++y)                       ... ++.2
```

For example:

$[1,2] ++ [3,4] = [1,2,3,4]$.

The "reverse" function yields the list whose elements follow the reverse order in respect to the argument list. The "reverse" function may be defined as follows:

```
reverse [] = []                              ... reverse.1
reverse a:x = (reverse x)++ [a]              ... reverse.2
```

For example:

reverse $[1,2,3,4] = [4,3,2,1]$.

The "sort" function, on lists whose elements are numeric atoms, yields the list whose elements do not follow in decreasing order.

The "sort" function may be defined as follows:

```
sort [] = []                                 ... sort.1
sort(a:x) = insert a (sort x)                ... sort.2
insert a []= [a]                             ... insert.1
insert a (b:x) = if a ≤ b then a:b:x
                 else b: insert a x          ... insert.2
```

In [6] a wrong definition of the "sort" function is given. The algorithm used is a particular method of sorting, usually called "insert-sort".

STRUCTURAL INDUCTION

To make formal proofs about list-like objects, which can be described by abstract rules, requires the use of structural induction. This is a technique proposed by R.M. Burstall ([11]) for dealing with trees of unbounded size.

The principle of structural induction for lists can be stated as follows. To prove that all lists have the property P, we show:

(a) P([]) and

(b) P(a:x) under the assumption P(x).

To say it differently,

(a) Prove the required property for the basic elements (empty list)

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(b1) assume the required property as an induction hypothesis for elements of the non-basic elements (the non empty list)

(b2) prove the required property for objects which contain previous elements.

Of course, this can be explained by an induction to the length of the list.

Let us prove by the structural induction the following lemma:

LEMMA 1. *A function* length x *is defined for every list* x .

P r o o f by induction on x

case [ ]

length [ ] = 0 by length.1

case (a:x)

length (a:x) = 1+length x by length.2 .

Since length x is ex hypothesi defined, then length (a:x) is defined too, and the lemma is proved.

Analogously, we can prove, by induction on x, that the functions cons a x, reverse x, insert a x and sort x are defined for every list x.

SOME PROPERTIES OF THE FUNCTION LENGTH

LEMMA 2. length $[a_1,\ldots,a_n]$ = n $(n \geq 1)$.

P r o o f.

$$
\begin{aligned}
\text{length}\,[a_1,\ldots,a_n] &= \text{length}\,(a_1:[a_2,\ldots,a_n]) && \text{by :}\\
&= 1+\text{length}\,[a_2,\ldots,a_n] && \text{by length.2}\\
&= 1+\text{length}\,(a_2:[a_3,\ldots,a_n]) && \text{by :}\\
&= 1+1+\text{length}\,[a_3,\ldots,a_n] && \text{by length.2}\\
&\;\;\vdots\\
&= n-1+\text{length}\,[a_n]\\
&= n-1+\text{length}\,(a_n:[\,]) && \text{by :}\\
&= n-1+1+\text{length}\,[\,] && \text{by length.2}\\
&= n+0 && \text{by length.1}\\
&= n.
\end{aligned}
$$

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LEMMA 3. length $x \geq 0$.

Proof by induction on x

case $[\,]$

length $[\,] = 0$ by length.1

case (a:x)

length (a:x) = 1+length x by length.2

$\geq$ 1+0 ex hypothesi

$\geq$ 0

as required.

THEOREM 1. length $x \to \infty$ *iff* $x \in \Omega$.

Proof. Let $x = [a_1, a_2, \ldots, a_n]$ be an Ω-list. Then:

length x = length $[a_1, a_2, \ldots, a_n, \ldots]$

= length $(a_1 : [a_2, \ldots, a_n, \ldots]$ by

= 1+length $([a_2, \ldots, a_n, \ldots])$ by length.2

. . .

=n+length $[a_{n+1}, \ldots,] \to \infty$.

Let length $x \to \infty$. By lemma 2 x can not be a finite list, then $x \in \Omega$.

Notice that, according to Theorem 1, the function length characterizes all the Ω-lists, i.e. lists which contain an infinite number of elements.

LEMMA 4. length x++y = length x + length y .

Proof by induction on x.

case

length $([\,]$++y) = length y by ++.1

= 0+length y

= length $[\,]$+length y by length.1

case (a:x)

length ((a:x)++y) = length (a:(x++y)) by ++.2

= 1+length (x++y) by length.2

=1+length x + length y ex hypothesi

= length (a:x) + length y by length.2

as required.

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LEMMA 5. $\text{length}(x{+}{+}y) \geq \text{length } x$,
$\text{length}(x{+}{+}y) \geq \text{length } y$.

P r o o f. Immediately follows from Lemma 3 and Lemma 4.

STRUCTURAL INDUCTION AND Ω-LISTS

We can formulate the following induction principle: to prove that a property P holds for all Ω-lists it is enough to prove:

(1) $P(\Omega)$, and

(2) $P(a{:}x)$ under the assumption $P(x)$.

This is an extension of the idea of structural induction, originally suggested in |6|, and is calles the partial object induction.

THEOREMA 2. $x{+}{+}y = x$ *for any* Ω*-list* x *(and arbitrary* y *)*.

P r o o f by a partial object induction on x:

case Ω

$\Omega{+}{+}y = \Omega$

Let us prove that the concatenation of an Ω-list x and arbitrary list y is an Ω -list. By Theorem 1, it suffices to prove that $\text{length}(\Omega{+}{+}y) \to \infty$.

Let $x = [a_1, a_2, \ldots]$ be an Ω-list, i.e. $x \in \Omega$.

$\text{length}(x{+}{+}y) = \text{length}([a_1, a_2, \ldots] {+}{+}y)$

$= \text{length}((a_1 : [a_2, \ldots]){+}{+}y)$ by:

$= \text{length}(a_1 : ([a_2, \ldots]{+}{+}y))$ by ++.2

$= 1{+}\text{length}([a_2, \ldots]{+}{+}y)$ by length.2

$= \ldots$

$= n{+}\text{length}([a_{n+1}, \ldots]{+}{+}y) \to \infty$

case $(a{:}x)$

$(a{:}x){+}{+}y = a{:}(x{+}{+}y)$ by ++.2

$= a{:}x$ ex hypothesi

as required.

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COROLLARY 1. *The concatenation of the two lists is a finite list iff both lists are finite lists.*

P r o o f. Follows from Lemma 5 and Theorem 2.

SOME PROPERTIES OF Ω-LISTS

By applying the function length it is possible to find out whether the value of a function, one of whose arguments is an Ω-list, is an Ω-list itself.

THEOREM 3. length(reverse x) = length x.

P r o o f by induction on x:

case [ ]

length(reverse[ ]) = length [ ] by reverse.1

case (a:x)

length(reverse(a:x)) = length(reverse x++[a]) by reverse.2
= length(reverse x)+length[a] by Lemma
= lengthx+length(a:[ ]) ex hypothesi; by
= lengthx+1+length [ ] by length.2
= lengthx+1+0 by length.1
= lengthx+1
= length(a:x) by length.2

as required.

COROLLARY 2. reverse Ω = Ω.

Let x∈Ω . If reverse x is a finite list then the value of the length (reverse x) is a finite number, which is impossible by Theorem 3.

LEMMA 6. length(insert a x) = 1 + length x .

P r o o f by induction on x

case [ ]

length(insert a [ ]) = length[a] by insert.1
= length (a:[ ]) by:
= 1+length [ ] by length.1

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case (b:x)

I let a ≤ b

length(insert a(b:x)) = length(a:b:x) by insert.2
= 1+length(b:x) by length.2

II Let a > b

length(insert a(b:x)) = length(b:insert a x) by insert.2
= 1+length(insert a x) by length.2
= 1+1+length x ex hypothesi
= 1+length(b:x) by length.2
as required.

THEOREM 4. length(sort x) = length x .

P r o o f by induction on x

case [ ]

length(sort[ ]) = length [ ] by sort.1

case (a:x)

length(sort(a:x)) = length(insert a(sort x)) by sort.2
= 1+length(sort x) by Lemma 6
= 1+length x ex hypothesi
= length(a:x) by length.2
as required.

COROLLARY 3. Sort Ω = Ω .

The list obtained by sorting an Ω-list is also an Ω - list.

ω- LISTS

A sublist is a non-empty tail of a non-empty

a) start list or

b) sublist.

For example:

the list [1[2[3[4]]]] has theree sublists:

[2[3[4]]]

[3[4]] and

[4]

The start list and empty list are not sublists.

The number of sublists of a list is equal to the num-

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ber of [, in the notation of the list, minus one.

An ω-list is the list whose notation is infinite, i.e. the list itself or some of its sublists has an infinite number of elements, or the number of sublists is infinite.

Obvious, Ω is a subset of ω .

For example:

$u = [\,[a_1,\ldots,a_n,\ldots]\,]$,

$v = v{:}[a] = [[[\ \ldots\ a]a]a]$

are ω-lists, i.e. they are elements of the set ω . v is a circular list and satisfies the following equations:

hd v = v and tl v = [a].

Lists v is an example of ω-lists , which can be implemented. In the general case the infinite lists can not be impleemented. For lists u and v holds:

length u = 1, and

length v = 2.

This shows that the function "length" can not characterize ω-lists.

Let us define the function len x in the following original way:

len [ ] = 0 ... len.1

len a = 0; atom a ... len.2

len (a:x)=1+len a + len x ... len.3

The function atom x is a basic function. The value of the function atom x is "true" iff x is not a list, otherwise it is "false".

LEMMA 7. $\mathrm{len}[a_1,\ldots,a_n] = n$ if $a_1,\ldots,a_n$ are atoms.

P r o o f by induction on n

case n = 0

len [ ] = 0 by len.1

case n > 0

$\mathrm{len}[a_1,\ldots,a_n] = \mathrm{len}(a_1:[a_2,\ldots,a_n])$ by:

$= 1+\mathrm{len}\ a_1+\mathrm{len}[a_2,\ldots,a_n]$ by len.3

$= 1+0+\mathrm{len}[a_2,\ldots,a_n]$ by len.2

$= 1+0+n-1$ ex hypothesi

$= n$

as required.

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LEMMA 8. *The value of the function* len x *is equal to the sum of the number of atoms of the list* x *and the number of sublists of the list* x .

P r o o f by induction on the number of sublists of list x

case n=0

Since there is no sublist, the proof immediately follows by Lemma 7.

case n > 0

Let $x = [a_1,\ldots,a_k]$

$\text{len } x = \text{len } [a_1,\ldots,a_k]$

$= \text{len}(a_1:[a_2,\ldots,a_k])$ by:

$= 1+\text{len } a_1+\text{len}[a_2,\ldots,a_k]$ by len.3

$= \ldots$

$= (1+\text{len } a_1)+(1+\text{len } a_2)+\ldots+(1+\text{len } a_k)$.

If a_i $(1\leq i\leq k)$ is an atom then the value of the expression $1+\text{len } a_i$ is equal to 1.

If a_i $(1\leq i\leq k)$ is a sublist then a unity in the expression $1+\text{len } a_i$ corresponds the sublist a_i $(1 \leq i \leq k)$. The number of sublists of the list a_i $(1 \leq i \leq k)$ is less than n, so $\text{len } a_i$ is equal to the total number of atoms and sublists of a_i (ex hypothesi). Since a_i was an arbitrary sublist, Lemma 8 holds.

THEOREM 5. $\text{len } x \to \infty$ *iff* $x \in \omega$.

P r o o f. Let $\text{len } x \to \infty$. If the number of sublists of list x is infinite then $x \in \omega$ by definition of ω. If the number of sublists of a list is finite then by Lemma 8 there is a sublist which is an Ω-list, so $x \in \omega$.

Let $x \in \omega$. Then the number of sublists or the number of elements of some sublist is infinite. In both cases by Lemma 8 $\text{len } x \to \infty$.

Between functions length and len there exists a simple relation.

THEOREM 6. $\text{length } x \leq \text{len } x$.

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Proof by induction on x

case []

length[] = len [] = 0 by length.1 and len.1

case (a:x)

length(a:x) = 1+length x by length.2
≤ 1+len x ex hypothesi
≤ 1+len x + len a
= len (a:x) by len.3
as required.

THEOREM 7. $x \in \Omega \Rightarrow x \in \omega$.

Proof directly follows by Theorem 6, 5 and 1.

SOME PROPERTIES OF ω-LISTS

Let us see for example function hd x. Let x = a:y .

LEMMA 9. len x > len(hd x)

Proof. len x = len (a:y)
= 1+len a + len y by len.3
> len a
= len(hd x).

LEMMA 10. hd $x \in \omega \Rightarrow x \in \omega$.

Proof is obvious. What can we conclude about Ω?

Let $a_1 \notin \Omega$ and $\underline{u} = [[a_1,\ldots,a_n,\ldots]]$. Then:

hd u = hd$[[a_1,\ldots,a_n,\ldots]]$
= $[a_1,\ldots,a_n,\ldots] \in \Omega$
hd hd u = hd$[a_1,\ldots,a_n,\ldots]$
= $a_1 \notin \Omega$.

Let $\underline{w} = [[a_1,\ldots,]b_1,\ldots]$. Then:

$\underline{w} \in \Omega$

hd w = hd$[[a_1,\ldots],b_1,\ldots]$
= $[a_1,\ldots] \in \Omega$.

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Consequently, all four cases, depending on whether the argument and the value of the function belong to Ω , are possible.

Let us see the function cons x y. It makes it easier to prove the following:

LEMMA 11. length(cons x y) = 1+length y ;
len(cons x y) = 1+len x + len y.

Consequently, function cons x y is an element of $\Omega(\omega)$ iff the least one of the arguments x and y is an element $\Omega(\omega)$.

LEMMA 12. len(x++y) = len x + len y.

P r o o f by induction on x

case [ ]

len ([ ]++y) = len y by ++.1
= len[ ] + len y by len.1

case (a:x)

len((a:x)++y) = len(a:(x++y)) by ++.2
= 1+len a + len(x++y) by len.3
= 1+len a + len x + len y by Lemma 12
= len(a:x) + len y by len.3
as required.

THEOREM 8. ω ++y = ω

P r o o f. Let x $\in \omega$. If x $\in \Omega$ then x++y $\in \Omega$ and x++y$\in \omega$. Otherwise $x = [a_1,\ldots,a_n]$ and len $x \to \infty$
By Lemma 12 follows:
len(x++y) = len x + len y $\to \infty$
then x++y $\in \omega$.

Remark that from x $\in \omega$, it is not possible to prove by structural induction that x++y = x.

LEMMA 13. len(reverse x) = len x

P r o o f by induction on x

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case [ ]

len(reverse[ ]) = len [ ] by reverse.1

case (a:x)

len(reverse a:x) = len(reverse x++[a]) by reverse.2
= len(reverse x)+ len[a] by Lemma 12
= len x + len[a] ex hypothesi
= len(a:x)
as required.

THEOREM 9. reverse ω = ω .

P r o o f immediately follows from Lemma 13 and Theorema 5.

LEMMA 14. len(insert a x) = 1 + len a + len x

P r o o f by induction on x

case [ ]

len(insert a[ ]) = len[a] by insert.1
= len(a:[ ]) by:
= 1+len a + len[ ] by len.3

case (b:x)

I a ≤ b

len(insert a b:x) = len(a:b:x) by insert.2
= 1+len a + len(b:x) by len.3

II a > b

len(insert a b:x) = len(b:insert a x) by insert.3
= 1+len b + len(insert a x) by len.3
≡ 1+len b+1+len a + len x ex hypothesi
= 1+len a + len(b:x) by len.3
as required.

LEMMA 15. len(sort x) = len x

P r o o f by induction on x

case [ ]

len(sort[ ]) = len[ ] by sort.1

case (a:x)

len(sort a:x) = len(insert a(sort x)) by sort.2
= 1+len a + len(sort x) by Lemma 14
= 1+len a + len x ex hypothesi

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= len(a:x)

as required.

THEOREM 10. sort(ω) = ω

P r o o f immediately follows from Lemma 15 and Theorem 5.

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REZIME

DOKAZIVANJE RAZNIH OSOBINA LISTI POMOĆU STRUKTURNE INDUKCIJE

U radu su definisani skupovi Ω i ω .

a) Ω je skup listi oblika

$[e_1, e_2, \ldots, e_n, \ldots]_{n \to \infty}$.

Svaki element e_n može biti:

- atom
- konačna lista

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- element skupa Ω
- element skupa ω

b) ω je skup listi koje ispunjavaju jedan od sledećih uslova:

- pripadaju skupu Ω
- neka njena podlista pripada skupu Ω
- sadrže beskonačno mnogo podlista.

Definisane su funkcije length i len na sledeći način:

length [] = 0
len (a:x) = 1+ length x
len [] = 0
len a = 0; atom a
len (a:x) = 1 + len a + len x

Dokazano je da važi:

length $x \to \infty \Rightarrow x \in \Omega$ i
len $x \to \infty \Leftrightarrow x \in \omega$.

Funkcije length i len se koriste za ispitivanje da li je vrednost funkcije čiji su elementi argumenti skupa Ω odnosno ω, element skupa Ω odnosno ω.

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SOME ALGEBRAIC PROPERTIES OF REGULAR MATROIDS

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ABSTRACT

The regular matroids mark an interesting half-way stage between the matroids corresponding to graphs on the one hand, and the binary matroids, that is matroids which are representable over GF(2), on the other. Perhaps the most famous result to date in all of matroid theory is Tutte's characterization of regular matroids by means of forbidden minors [2]. An interesting feature of regular matroids is their close relationship with an important class of matrices, the unimodular matrices [4] (note that the entries of a unimodular matrix are all 0 or ± 1).

Our aim in this paper is to give some algebraic properties of the standard representatives matrices of regular matroids.

INTRODUCTION

The matroid theoretic terminology and results used in this paper are according to standard literature (e.g., see [2, 4,5]). Let E be a finite set and r a function $r : 2^E \to \mathbb{N}$ (2^E is the power set of E and $\mathbb{N}$ the set of non-negative integers). Then the pair (E,r) is a matroid $M := M(E,r)$ on E, and r(S) is the rank of $S \subseteq E$, if the following conditions hold:

(a) $r(S) \leq |S|$, for each $S \subseteq E$,

(b) if $S \subseteq S' \subseteq E$, then $r(S) \leq r(S')$,

(c) $r(S \cup S') + r(S \cap S') \leq r(S) + r(S')$, for each $S, S' \subseteq E$.

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A subset $S \subseteq E$ is called independent if $r(S) = |S|$ where $|S|$ denotes the cardinality of S; a basis of M is a maximal independent subset of E.

If B is a basis of M, then $B^* = E - B$ is called a cobasis of M. The dual matroid of M is the matroid M^* on E whose bases are the cobases of M. If r^* is the rank function of M^*, then $r^*(S) = |S| - r(E) + r(E-S)$ for every $S \subseteq E$. Let $F(M)$ be the family of independent sets of M, and $\mathbb{F}$ a field. M is representable |1| over $\mathbb{F}$ if there exists a vector space V over $\mathbb{F}$ and an injection $\sigma : E \to V$ such that a subset S of E belongs to $F(M)$ if and only if the corresponding vectors of $\sigma(S)$ are linearly independent over $\mathbb{F}$. A matroid is regular |4| if it is representable over any field.

Throughout, we shall denote $r := r(E)$; it is well-known that for any basis B of M we have $r(B) = r(E)$. Let then B be a basis of M and $m = |E-B|$. If M is representable over a field $\mathbb{F}$ it will have a standard matrix representation |2,4| with respect to the basis B of the form $R(M,B) = [I_r | A]$ where I_r is the $r \times r$ identity matrix and A is an $r \times m$ matrix with the entries belonging to $\mathbb{F}$.

A well-known property of matroid representation (stated by Tutte in |2|) says that if M has a standard representation $R(M,B) = [I_r | A]$, then the dual M^* has a standard representation $R^*(M,B^*) = [-A^T | I_m]$ where A^T is the transposed of A and $B^* = E - B$. The following hold:

$$(1) \qquad [R(M,B)]\,[R^*(M,B^*)]^T = O_{rm},$$

$$(2) \qquad [R^*(M,B^*)]\,[R(M,B)]^T = O_{mr},$$

where O_{pq} denotes the null matrix with p rows and q columns.

The main results. In the sequel we shall consider M to be a regular matroid on the finite set $E = \{e_1, e_2, \ldots, e_n\}$. If S is any subset of E and R a standard representation matrix of M we define R(S) as the submatrix of R consisting of those columns that correspond to members of S.

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THEOREM 1. *(W.T.Tutte, |3|). The matrices* R(M,B) *and* $R^*(M,B^*)$ *are unimodular.*

THEOREM 2. *(W.T.Tutte, |3|). Let* S *be a subset of* E. *The determinant of* R(S) *has one of the values* 1 *or* -1 *if* S *is a basis of* M *and* 0 *otherwise.*

THEOREM 3. *(W.T.Tutte, |3|). Let* S *be a subset of* E. *The determinant of* R*(S) *has one of the values* 1 *or* -1 *if* S *is a cobasis of* M *and* 0 *otherwise.*

COROLLARY *(W.T.Tutte, |3|). The following hold:*

$$\det\left(\left[R(M,B)\right]\left[R(M,B)\right]^T\right) = b(M)\,, \tag{3}$$

$$\det\left(\left[R^*(M,B^*)\right]\left[R^*(M,B^*)\right]^T\right) = b(M)\,. \tag{4}$$

P r o o f. It follows from Theorem 1 and 2 by applying the Binet-Cauchy theorem:

$$\det\left(\left[R(M,B)\right]\left[R(M,B)\right]^T\right) = \sum_{i=1}^{b(M)} \left[\det R(B_i)\right]^2 = \sum_{i=1}^{b(M)} 1 = b(M),$$

where B_i, i=1,2,...,b(M) are the bases of M. Similarly we can prove (4). (Q.E.D.)

LEMMA 1. *For any* r x m *matrix* A *the following hold:*

$$\det(I_r + AA^T) \geq 0.$$

P r o o f. Let X be an arbitrary r x 1 vector. Thus $X^T(I_r + AA^T)X = X^TX + X^TAA^TX = X^TX + (A^TX)^T(A^TX) = ||X||^2 + ||A^TX||^2 \geq 0$, i.e., $I_r + AA^T$ is a positive definite matrix ($||X||$ denotes the norm of X). Since $I_r + AA^T$ is symmetrical it follows by Sylvester's criterion (e.g., see |6|) that $\det(I_r + AA^T) \geq 0$. (Q.E.D.)

LEMMA 2. |6| *If* Y *is a square matrix of the form* $Y = \left[\begin{array}{c|c} A & B \\ \hline C & D \end{array}\right]$ *where* A *and* D *are square matrices, then the following hold:*

(a) *if* $\det A \neq 0$, *then* $\det Y = \det A \det(D - CA^{-1}B)$,

(b) *if* $\det D \neq 0$, *then* $\det Y = \det D \det(A - BD^{-1}C)$.

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LEMMA 3. *The following hold:*

$$(c) \qquad \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] \geq 0 \ ,$$

$$(d) \qquad \det\left[\frac{R^*(M,B^*)}{R(M,B)}\right] = (-1)^{mr} \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] .$$

P r o o f. According to Lemma 2 we have

$$\det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] = \det\left[\begin{array}{c|c} I_r & A \\ \hline -A^T & I_m \end{array}\right] = \det(I_m + A^T A) = \det(I_r + AA^T),$$

and (c) follows from Lemma 1. On the other hand, by Lemma 2, we have

$$\det\left[\frac{R^*(M,B^*)}{R(M,B)}\right] = \det\left[\begin{array}{c|c} -A^T & I_m \\ \hline I_r & A \end{array}\right] =$$

$$= (-1)^{mr} \det\left[\begin{array}{c|c} I_m & -A^T \\ \hline A & I_r \end{array}\right] = (-1)^{mr} \det\,(I_r + AA^T) =$$

$$= (-1)^{mr} \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] \quad . \quad \text{(Q.E.D.)}$$

THEOREM 4. *The following hold:*

$$(5) \qquad \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] = b(M)$$

$$(6) \qquad \det\left[\frac{R^*(M,B^*)}{R(M,B)}\right] = \pm\, b(M) \ .$$

P r o o f. From (1) - (4) we obtain:

$$\left(\det\left[\frac{R(M,B)}{R^*(M,B^*)}\right]\right)^2 = \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] =$$

$$= \det\left[\frac{R(M,B)}{R^*(M,B^*)}\right] \det\left[\begin{array}{c|c} [R(M,B)]^T & [R^*(M,B^*)]^T \end{array}\right] =$$

$$= \det\left(\left[\frac{R(M,B)}{R^*(M,B^*)}\right] \left[\begin{array}{c|c} [R(M,B)]^T & [R^*(M,B^*)]^T \end{array}\right]\right) =$$

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$$= \det \left[\begin{array}{c|c} [R(M,B)]\,[R(M,B)]^T & [R(M,B)]\,[R^*(M,B^*)]^T \\ \hline [R^*(M,B^*)]\,[R(M,B)]^T & [R^*(M,B^*)]\,[R^*(M,B^*)]^T \end{array} \right] =$$

$$= \det \left[\begin{array}{c|c} [R(M,B)]\,[R(M,B)]^T & O_{rm} \\ \hline O_{mr} & [R^*(M,B^*)]\,[R^*(M,B^*)]^T \end{array} \right] =$$

$$= (-1)^{2\frac{r(r+1)}{2}} \det([R(M,B)]\,[R(M,B)]^T) \det([R^*(M,B^*)]\,[R^*(M,B^*)]^T) =$$

$$= [b(M)]^2 .$$

Thus, (5) follows according to (c) and (6) follows by (5) and (d). (Q.E.D.) Let B be a fixed basis of $M, B^* = E - B$ and B_i, $B_i^* = E - B_i$, $i=1,2,\ldots,b(M)$. For every B_i we denote by $s(B_i)$ the sum of the columns´ indices of R(M,B) that correspond to members of B_i. Similarly for $s(B_i^*)$. By expansion of $\det\left[\frac{R(M,B)}{R^*(M,B^*)}\right]$ according to Laplace´s rule considering all major square submatrices of order r contained in the rows of R(M,B), and using Theorem 2 and 3 we obtain from (5):

$$S = \sum_{i=1}^{b(M)} (-1)^{s(B_i)} \det R(B_i) \det R^*(B_i^*) = b(M) .$$

Similarly, using R*(M,B*), by (6) we have:

$$S^* = \sum_{i=1}^{b(M)} (-1)^{s(B_i^*)} \det R(B_i) \det R^*(B_i^*) = \pm b(M) .$$

Obviously, $S = b(M)$ if and only if $(-1)^{s(B_i)} \det R(B_i) \det R^*(B_i^*) = 1$ for every $i=1,2,\ldots,b(M)$, i.e. if and only if $\det R(B_i) = (-1)^{s(B_i)} \det R^*(B_i^*)$ for every $i=1,2,\ldots,b(M)$. Similarly $S^*=b(M)$ if and only if $\det R(B_i) = (-1)^{s(B_i^*)} \det R^*(B_i^*)$ for every $i=1,2,\ldots,b(M)$. On the other hand, $S^* = -b(M)$ if and only if $(-1)^{s(B_i^*)} \det R(B_i) \det R^*(B_i^*) = -1$ for every $i=1,2,\ldots,b(M)$, i.e., if and only if $\det R(B_i) = (-1)^{s(B_i^*)+1} \det R^*(B_i^*)$ for every $i=1,2,\ldots,b(M)$. Thus we have the following theorems:

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THEOREM 5. *The following holds:*

$$\det R(B_i) = (-1)^{s(B_i)} \det R^*(B_i^*), \quad i=1,2,\ldots,b(M).$$

THEOREM 6. *Only one of the following holds:*

either $\det R(B_i) = (-1)^{s(B_i^*)} \det R^*(B_i^*)$, $i=1,2,\ldots,b(M)$,

or $\det R(B_i) = (-1)^{s(B_i^*)+1} \det R^*(B_i^*)$, $i=1,2,\ldots,b(M)$.

Let B' be a basis of M such that $B \neq B'$, $B - B' = \{e_{i_1}, e_{i_2},\ldots,e_{i_t}\}$ and $B' - B = \{e_{j_1}, e_{j_2},\ldots,e_{j_t}\}$ (obviously $|B-B'| = |B'-B|$). We shall denote by $R(B-B', B'-B)$ the square submatrix of $R(M,B)$ consisting of elements in the crossing of the rows $i_1, i_2,\ldots,i_t$ with the columns $j_1, j_2,\ldots,j_t$.

THEOREM 7. *The following holds:*

$$\det R(B-B', B'-B) = \det R(B') .$$

P r o o f. Obviously, the columns $t+1, t+2,\ldots,r$ of $R(M,B)$ have only one non-null entry, i.e.,

$$(7) \quad (R(M,B))_{t+1,t+1} = (R(M,B))_{t+2,t+2} = \ldots = (R(M,B))_{r,r} = 1.$$

Let Δ_{t+1} denote the subdeterminant of $\det R(B')$ obtained by deleting the row $t+1$ and the column $t+1$: Δ_{t+2} the subdeterminant of Δ_{t+1} obtained by deleting the row $t+2$ and the column $t+2$ and so on up to Δ_r. From (7) and the definition of $R(M,B)$ it follows that $\det R(B') = \Delta_{t+1} = \Delta_{t+2} = \ldots = \Delta_r = \det R(B-B', B'-B)$. (Q.E.D.).

Similarly, if $B^{*\prime}$ is a cobasis of M such that $B^{*\prime} \neq B^*$ we then have:

THEOREM 8. *The following holds:*

$$\det R^*(B^* - B^{*\prime}, B^{*\prime} - B^*) = \det R^*(B^{*\prime}).$$

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Let B_a, B_b be two distinct bases of M, B_a^*, B_b^* their corresponding cobases and R_a, R_b, R_a^*, R_b^* the respectively standard representative matrices. Let $d(B_a,B_b) = |B_a - B_b| = |B_b - B_a|$. According to the form of R and R* and since $B_a - B_b = B_b^* - B_a^*$, $B_b - B_a = B_a^* - B_b^*$ we then have

$$(8) \quad \det R_a(B_a - B_b,\ B_b - B_a) = (-1)^{d(B_a,B_b)} \det R_a^*(B_a^* - B_b^*, B_b^* - B_a^*) ,$$

$$(9) \quad \det R_b(B_b - B_a, B_a - B_b) = (-1)^{d(B_a,B_b)} \det R_b^*(B_b^* - B_a^*, B_a^* - B_b^*) .$$

From (8) and (9), using Theorems 7 and 8 we obtain:

$$(10) \quad \det R_a(B_b) = (-1)^{d(B_a,B_b)} \det R_a^*(B_b^*) ,$$

$$(11) \quad \det R_b(B_a) = (-1)^{d(B_a,B_b)} \det R_b^*(B_a^*) .$$

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REZIME

NEKE ALGEBARSKE OSOBINE REGULARNIH MATROIDA

U ovom radu dokazane su neke algebarske osobine standardnih reprezentativnih matrica regularnih matroida.

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ANOTHER CONSTRUCTION OF RANK 4 PAVING MATROIDS ON 8 ELEMENTS (I)

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ABSTRACT

All 950 non-isomorphic simple matroids on 8 elements were constructed in [2] by the use of a computer programme. By using elementary methods, without computer aid, we give another construction for probably the most interesting (when non-isomorphisms are considered) subclass P of 322 rank 4 paving matroids on 8 elements. The class P is partitioned into three disjoint subclasses. The construction of the first two is given in this paper, while the construction of the third subclass of P is given in a sequel paper [1] (these two papers make a whole). We study in greater detail the ways in which the non-isomorphic possibilities arise. Our main tool in the construction of P are three auxiliary classes of graphs (these graphs can be bijected to some of the paving matroids) and some properties of the Steiner system $S(3,4,8)$.

PRELIMINARIES

We emphasize that the whole understanding of the construction itself can be gained without knowing what a matroid is, although some familiarity with elements of matroid theory is desirable. A few basic graph-theoretical definitions are necessary; all can be found in any general text on graphs (e.g., [3]). Non-defined notions on matroids (which appear only in several commentaries of the construction) can be found in [4].

We shall mainly consider the sets which are subsets of the set $S = \{1,2,3,4,5,6,7,8\}$. If there is no possibility of

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confusion, then the fixed subsets of S are denoted without brackets and commas (e.g., "1234" instead of "{1,2,3,4}"). We shall often speak about, e.g., "1234" instead of "the set 1234". An n-set (n-intersection, etc,) is a set (intersection, etc.) of cardinality n. In particular, an n-path of a graph is a path of n (distinct) edges. The cardinality of a set X is denoted by $|X|$.

By a "graph" we shall always mean a non-oriented graph without loops and multiple edges.

A ground-set of a family of sets is the union of all its sets.

Two families of sets F_1 and F_2, with the ground-sets G_1 and G_2 respectively, are isomorphic if there is a bijection $\alpha : G_1 \to G_2$ such that

$$X \in F_1 \iff \alpha(X) \in F_2$$

The addition of a set X to a family F, where X does not belong to F, is the operation which gives the family $F \cup \{X\}$. The set X is said to be added to F.

If the addition of several different sets to the same family F gives the isomorphic families with one set more, then we choose one of these sets. Its addition to F gives a representative of a class of isomorphic larger families.

A P-family is a family F of distinct subsets of S satisfying:

(a) $X \in F \Rightarrow |X| \geq 4$

(b) $(X_1, X_2 \in F \wedge X_1 \neq X_2) \Rightarrow |X_1 \cap X_2| \leq 2$

(c) $S \notin F$

An A-family is a P-family F which additionally satisfies:

(d) $(\exists X)(X \in F \wedge |X| \geq 5)$

A B-family is a family F of distinct 4-subsets of S satisfying:

(e) $(X_1, X_2 \in F \wedge X_1 \neq X_2) \Rightarrow |X_1 \cap X_2| \in \{0,2\}$

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A C-family is a P-family, which is neither an A-family nor a B-family.

INTRODUCTION

There is an obvious bijection between non-isomorphic P-families and non-isomorphic rank 4 paving matroids on 8 elements (P-families correspond exactly to the families of non-trivial hyperplanes of such matroids). On the other hand, it is clear that non-isomorphic P-families may be partitioned into non-isomorphic A-, B- and C-families.

In this paper we shall give separate constructions for all non-isomorphic A- and B-families. There are, respectively, 52 and 86 families in these classes. The construction of all non-isomorphic C-families is given in |1|.

Our main tool for dealing with the lion´s share of non-isomorphic A- and C-families are three classes of auxiliary graphs. We establish the theorems which provide the necessary correspodence in each case. On the other hand, the construction of all non-isomorphic subfamilies of the Steiner system S(3, 4,8) yields all the non-isomorphic B-families.

The non-isomorphism of any two constructed matroids (i.e., of any two constructed P-families) should be obvious from our construction. This is not so in catalogue |2|. For example, there are 63 non-isomorphic rank 4 paving matroids on 8 elements, which are given in |2| by families of seven 4-hyperplanes.

The exhaustion of all non-isomorphic possibilities immediately follows from |2|. We have made efforts, however, to provide a self-consistent proof of this exhaustion, although a full understanding of such a proof sometimes still requires small case analyses.

We would by no means recommend our construction instead of that of |2|:, in fact, our construction could be

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hardly completed without the help of the first one. We simply hope that this construction could help a better understanding of the ways in which non-isomorphic matroids arise and, perhaps, be used for some further research.

FURTHER DEFINITIONS AND DENOTATIONS

Given an A-family F, which has exactly one 5-set, denoted by X, we define the A-graph G of F as follows:

(i) vertices of G are elements of X

(ii) there is an edge $\{a,b\}$ in G if and only if there is a 4-set containing $\{a,b\}$ in F.

Two (non-incident) edges of the just defined A-graph G are said to be similar if the corresponding two 4-sets of F contain the same 2-subset of $S\setminus X$.

A denoted A-graph is a graph G on five vertices, with at most six edges, such that

(i) some disjoint pairs of non-incident edges may be distinguished

(ii) if G has k edges $(4 \leq k \leq 6)$, then there are at least k-3 pairs of distinguished edges .

REMARK: (ii) implies that G has not a vertex of degree 4.

The pairs of distinguished edges determine a denotation of G.

We stress that it should be proved that denoted A-graphs are A-graphs of some A-families. In fact, the definition of denoted A-graphs includes the characterization of those graphs, which can be represented as A-graphs of some A-families.

The family of blocks of the Steiner system $S(3,4,8)$ is the family ϕ, where:

$$\phi = \begin{Bmatrix} 1234, & 1256, & 1278, & 1357, & 1368, & 1458, & 1467 \\ 5678, & 3478, & 3456, & 2468, & 2457, & 2367, & 2358 \end{Bmatrix}$$

(The blocks are for convenience written in two lines).

The complement of a block X of ϕ is the block $S\setminus X$.

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The complementary subfamily of a subfamily F of ϕ is the family $\bar{F}$ such that $F \cup \bar{F} = \phi$; $F \cap \bar{F} = \emptyset$.

The family ϕ_1 is the family of those blocks of ϕ, which contain the element 1 (the first row in the entry above). The blocks of ϕ without 8 are called heptahedron hyperplanes.

A complement of a subfamily F of ϕ is a block, which is not in F, but the complement of which is in F. In some cases we say only "the complement", when F is clear from the context.

We shall sometimes speak about some elements or some subsets of S sharing the isomorphic position in (with respect to) a subfamily F of ϕ. We found it difficult, and perhaps unnecessary in this context, to make this rather vague notion quite precise. It means that the corresponding subsets of S should be equally treated when constructing non-isomorphic subfamilies of ϕ, which contain F. Such subsets are easily recognized in each particular case. For example, a necessary condition for two elements of S to share the isomorphic position in F is for them to appear the same number of times in the blocks of F. In most cases this condition appears to be sufficient. A similar conclusion holds for 2-subsets of S.

A subset of S has a special position in a subfamily of ϕ if it does not share the isomorphic position (with respect to that subfamily) with another subset of S.

CONSTRUCTION OF NON-ISOMORPHIC A-FAMILIES

We differentiate four cases for an A-family F:

Case 1: F has a 7-set X.

The rules (a) and (b) give $F = \{X\}$.

Case 2: F has a 6-set X.

It is easily seen that all the other sets of F can be just the 4-sets which contain $S \setminus X$ and no two have a common element in X. There are therefore just four non-isomorphic

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A-families with a 6-set, having from none up to three 4-sets.

Case 3: F has two 5-sets X_1 and X_2.

We have $|X_1 \cap X_2| = 2$. An easy argument shows that all the other sets of F can be just the 4-sets having two elements in $X_1 \setminus X_2$ and the other two in $X_2 \setminus X_1$. At most three such 4-sets may appear simultaneously and each two of them have one common element in $X_1 \setminus X_2$ and another one in $X_2 \setminus X_1$. So again we have four non-isomorphic possibilities depending on the number of 4-sets.

Case 4: F has exactly one 5-set X.

If F has a 4-set containing $S \setminus X$, then we denote this 4-set by Y.

THEOREM 1.*) *There is a bijection between non-isomorphic*

a) *denoted A-graphs*	*and*	*A-families which have exactly one 5-set X and have not Y*
b) *denoted A-graphs which have at least one isolated vertex*	*and*	*A-families which have exactly one 5-set X and which have Y .*

These bijections are realized by establishing such isomorphisms between the denoted A-graphs and the A-graphs of the A-families, which map pairs of distinguished edges to pairs of similar edges and conversely.

P r o o f. (Sketch) An A-family F of Case 4. has the unique A-graph G with fixed pairs of similar edges. It is easily shown that G satisfies all the conditions to be a denoted A-graph, when the similar edges are interpreted as distinguished. On the other hand, a denoted A-graph uniquely determines, up to an obvious isomorphism, the corresponding A-family. As for b), we observe that the element $X \cap Y$ cannot appear in another 4-set of F.

We give the table of all non-isomorphic denoted A-graphs. The graphs are given without isolated vertices, if

*) Two denoted A-graphs G_1 and G_2 are isomorphic if there is a graph-isomorphism which maps G_1 onto G_2 and preserves the denotation.

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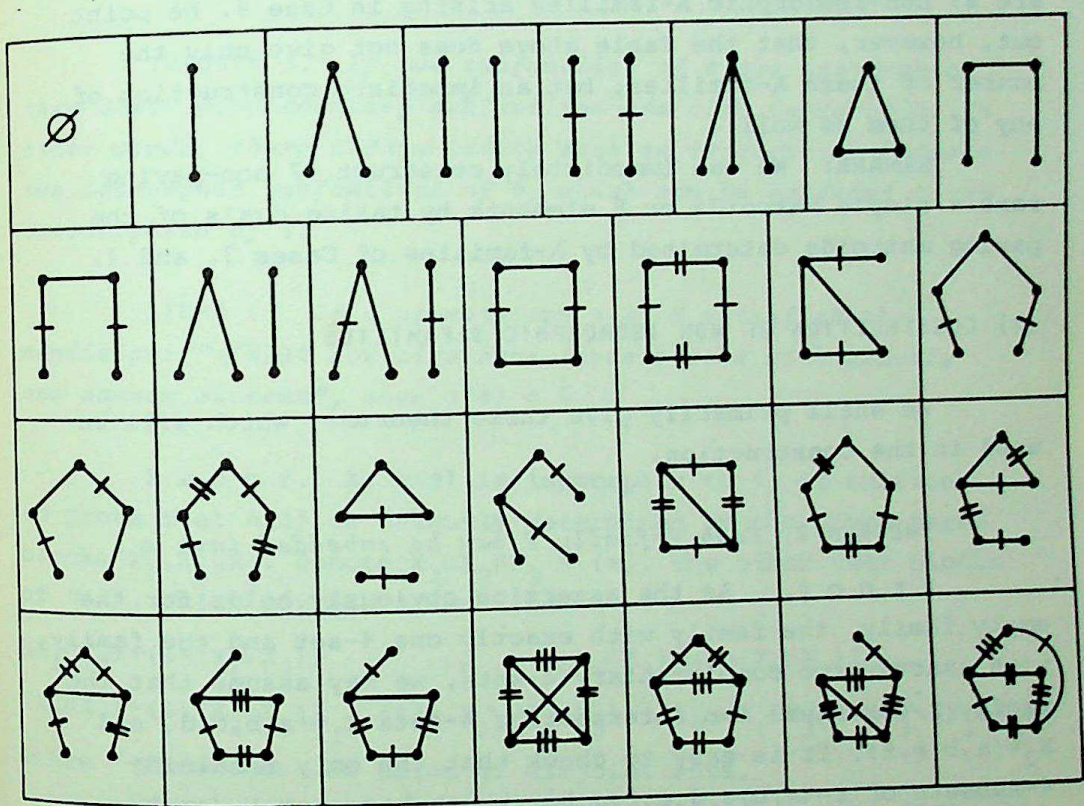
there are any. Two distinguished edges in the same pair are always crossed by the some number of short lines. The order of this denotation is unimportant.

THE TABLE OF NON-ISOMORPHIC DENOTED A-GRAPHS



It is well-known (see,e.g., the Appendix of |2|) that the above list includes (when the isolated vertices are returned) all non-isomorphic graphs on five vertices, which have at most six edges and which have not a vertex of degree 4. The exhaustion of non-isomorphic possibilities for denotation, however, should be checked by case analysis. Thus,

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for example, the last denoted A-graph in the table has two vertices of degree 3 connected by three disjoint paths of length 2. Any edge has just two non-incident edges and it is easy to see that there are only two isomorphic possibilities for denotation.

As there are 29 non-isomorphic denoted A-graphs, 14 of which have isolated vertices, Theorem 1 gives that there are 43 non-isomorphic A-families arising in Case 4. We point out, however, that the Table above does not give only the number of these A-families, but an immediate construction of any of them as well.

REMARK: We can immediately construct 47 non-paving rank 4 simple matroids on 8 elements by taking duals of the paving matroids determined by A-families of Cases 3. and 4.

(2) CONSTRUCTION OF NON ISOMORPHIC B-FAMILIES

We shall primarily give three theorems, which will be used in the construction.

THEOREM 2. *Each B-family F can be embedded into* ϕ.

P r o o f. As the assertion obviously holds for the empty family, the family with exactly one 4-set and the family with exactly two complementary 4-sets, we may assume that the family F contains two intersecting 4-sets $X_1=\{a,b,c,d\}$ and $X_2=\{a,b,e,f\}$. It is easy to check that the only remaining 4-subsets of $S=\{a,b,c,d,e,f,g,h\}$, which have not 1- or 3-intersections with either of X_1, X_2, are the sets of IUJUK, where

$$I=\{\{a,b,g,h\},\ \{c,d,e,f\},\ \{c,d,g,h\},\ \{e,f,g,h\}\}$$

$$J=\left\{\begin{matrix}\{a,c,e,g\},\ \{a,c,f,h\},\ \{a,d,e,h\},\ \{a,d,f,g\}\\ \{b,d,f,h\},\ \{b,d,e,g\},\ \{b,c,f,g\},\ \{b,c,e,h\}\end{matrix}\right\}$$

$$K=\left\{\begin{matrix}\{a,c,e,h\},\ \{a,c,f,g\},\ \{a,d,e,g\},\ \{a,d,f,h\}\\ \{b,d,f,g\},\ \{b,d,e,h\},\ \{b,c,f,h\},\ \{b,c,e,g\}\end{matrix}\right\}.$$

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There are no "forbidden" intersections among the sets of IUJ or IUK, while each set of J has either 1- or 3-intersection with each set of K. Thus the only two maximal B-families containing $X=\{X_1,X_2\}$ are $X+I+J = \Phi_2$ and $X+I+K = \Phi_3$. We have, however, that $\Phi_2 \simeq \Phi$ and $\Phi_3 \simeq \Phi$. The isomorphisms are established by the permutations, which map {a,b,c,d,e,f,g,h} to {1,2,3,4,5,6,7,8} and {1,2,3,4,5,6,8,7} respectively.

THEOREM 3. *If two subfamilies of Φ are isomorphic, then their complementary subfamilies are also isomorphic. In other words, there always exists such an isomorphism between two isomorphic subfamilies of Φ, which can be extended to an automorphism of Φ.*

LEMMA 1. *If a permutation α of S satisfies the condition: "$\alpha(\Phi)\cap\Phi$ contains some three blocks with exactly one common element", then $\alpha(\Phi) = \Phi$*

P r o o f. As $\alpha(\Phi)$ is isomorphic to Φ, it suffices to prove that $\alpha(\Phi)$ is uniquely determined by the given three blocks X_1,X_2,X_3. Denote $X_1\cap X_2\cap X_3 = \{z\}$. The other four blocks of $\alpha(\Phi)$, which contain z, must be:

$$(X_1\cap X_2)+(S\setminus(X_1\cup X_2)) \quad , \quad (X_1\cap X_3)+(S\setminus(X_1\cup X_3))$$

$$(X_2\cap X_3)+(S\setminus(X_2\cup X_3)) \quad , \quad \{z\}+(X_1\setminus(X_2\cup X_3))+(X_2\setminus(X_1\cup X_3))+(X_3\setminus(X_1\cup X_2))$$

where "+" denotes the union of disjoint sets.

Namely, the first three blocks are due to the fact that each of the 2-sets $X_1\cap X_2$, $X_1\cap X_3$, $X_2\cap X_3$ appears in exactly three blocks of $\alpha(\Phi)$. A similar argument applied to the 2-sets $\{z\}+(X_1\setminus(X_2\cup X_3))$, $\{z\}+(X_2\setminus(X_1\cup X_3))$, $\{z\}+(X_3\setminus(X_1\cup X_2))$ yields the fourth block.

The remaining seven blocks of $\alpha(\Phi)$ must be the complements of the first seven.

LEMMA 2. *If α is an isomorphism between two subfami-*

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lies F_1 and F_2 of Φ, each of which contains (at least) three blocks with exactly one common element, then α can be extended to an automorphism of Φ.

P r o o f. As the cardinality c of the ground-set of F_1 is at least 7, the bijection α between the ground-sets of F_1 and F_2 can be extended to a unique permutation $\bar{\alpha}$ of S (if c=8, then $\bar{\alpha}=\alpha$). The proof is completed by applying Lemma 1.

P r o o f of Theorem 3. Due to Lemma 2., we have to consider only the case when none of two isomorphic subfamilies F_1 and F_2 of Φ has three blocks with exactly one common element. We differentiate three cases:

Case 1. F_1 has two intersecting blocks

$$X_1 = \{a,b,c,d\} \quad , \quad X_2 = \{a,b,e,f\}.$$

The 3-set {a,c,e} is included in a block $X_3=\{a,c,e,g\}$ of Φ, for some $g\in S\setminus(X_1\cup X_2)$. Let α denote any isomorphism of F_1 onto F_2. Similarly there exists a block $X_4=\{\alpha(a), \alpha(c), \alpha(e), v\}$ of Φ, for some $v\in S\setminus(\alpha(X_1)\cup\alpha(X_2))$. Obviously $X_3\notin F_1, X_4\notin F_2$. We define a permutation α_1 of S as follows:

$$\alpha_1(x)=\alpha(x) \quad \text{for} \quad x\in X_1\cup X_2, \qquad \alpha_1(g)=v$$
$$\alpha_1(S\setminus(\{g\}\cup X_1\cup X_2)) = S\setminus(\{v\}\cup\alpha(X_1)\cup\alpha(X_2)).$$

The permutation α_1 estabilishes an isomorphism between the subfamilies $\{X_1,X_2,X_3\}$ and $\{\alpha(X_1),\alpha(X_2),X_4\}$ of Φ. Thus α_1 is an automorphism of Φ by Lemma 2.

The only blocks, which may arise in F_1 (by reason of the assumption), apart from X_1 and X_2, are:

$$(X_1\cup X_2)\setminus(X_1\cap X_2),\ (X_1\cap X_2)+(S\setminus(X_1\cup X_2)),\ S\setminus X_1,\ S\setminus X_2.$$

Since $\alpha_1(X_1)=\alpha(X_1)$ and $\alpha_1(X_2)=\alpha(X_2)$, the images of each of these blocks under α and α_1 coincide. This implies that α_1 is an extension of α.

Case 2. F_1 has a block $X_1=\{a,b,c,d\}$, but has not two intersecting blocks.

There exist two blocks $X_2=\{a,b,e,f\}$ and $X_3=\{a,c,e,g\}$

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of Φ for some $e,f,g \in S \setminus X_1$. $X_2, X_3 \notin \Phi_1$.

Let α denote an arbitrary isomorphism of F_1 onto F_2. Similarly, there exist blocks $X_4 = \{\alpha(a), \alpha(b), h, i\}$ and $X_5 = \{\alpha(a), \alpha(c), h, j\}$ of Φ for some $h,i,j \in S \setminus \alpha(X_1)$. Obviously $X_4, X_5 \notin F_2$. We define the following permutation α_2 of S:

$$\alpha_2(x) = \alpha(x) \quad \text{for } x \in X_1$$

$$\alpha_2(e) = h \quad ; \quad \alpha_2(f) = i \quad ; \quad \alpha_2(g) = j$$

$$\alpha_2(S \setminus (X_1 \cup X_2 \cup X_3)) = S \setminus (\alpha(X_1) \cup X_4 \cup X_5).$$

The permutation α_2 establishes an isomorphism between the subfamilies $\{X_1, X_2, X_3\}$ and $\{\alpha(X_1), X_4, X_5\}$ of Φ. α_2 is an automorphism of Φ by Lemma 2.

The only block, which may arise in F_1, apart from X_1, is $S \setminus X_1$ and

$$\alpha_2(S \setminus X_1) = S \setminus \alpha_2(X_1) = S \setminus \alpha(X_1) = \alpha(S \setminus X_1).$$

proves that α_2 is an extension of α.

Case 3. F_1 has no blocks

$\Phi \setminus F_1 = \Phi \setminus F_2 = \Phi$, which completes the proof of Theorem 3.

Consequence: Non-isomorphic subfamilies of Φ, which have more than seven blocks, are uniquely determined (up to an isomorphism) as the complementary subfamilies of those non-isomorphic subfamilies of Φ, which have at most six blocks.

THEOREM 4. *The assertion of Theorem 3 still holds, when Φ is replaced by Φ_1.*

P r o o f. Omitted as similar to and easier than the previous one. (Deleting 1 from all sets of Φ_1, we have the lines of the Fano plane, and we should use the families of three non-concurrent lines as the "key").

Theorem 2 and the Consequence of Theorem 3 reduce the construction of all non-isomorphic B-families to the construction of all non-isomorphic subfamilies of Φ which have at most

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seven blocks. This last construction will be done step-by-step, beginning with the smaller subfamilies. Given a subfamily of k blocks, we should naturally look for its non-isomorphic superfamilies of k+1 blocks and choose one representative from each class of isomorphic families for the next step of generation. A great defect of this procedure, however, is that non-isomorphic families may have isomorphic superfamilies. In order to lessen its effect as much as possible, we develop the following approach:

We shall favorize the element 1 of S in our construction in the following sense:

No element of {2,3,4,5,6,7,8} appears in more blocks than 1 does and no 2-subset of S appears in three blocks of Φ if a 2-subset containing 1 does not,

in any subfamily of Φ which we construct (to be the representative of a class of isomorphic subfamilies).

This condition still preserves the generality, because all the elements, respectively 2-subsets, of S are in isomorphic positions in Φ. What is more, we may assume that the 4-sets containing 1, of any subfamily of Φ that we construct, necessarily form one of the fixed (representatives of) non-isomorphic subfamilies of Φ_1.

We primarily construct non-isomorphic subfamilies of Φ_1 and add thereafter some 4-sets from $\Phi \setminus \Phi_1$ to them in order to obtain other subfamilies of Φ. For each number of blocks we primarily list the representatives of the corresponding non-isomorphic subfamilies (separated by commas). This list is, except for the trivial cases, followed by short explanations of the construction.

NON-ISOMORPHIC SUBFAMILIES OF Φ_1

0 blocks: $\emptyset$

1 block: A = {1234}

2 blocks: B = {1234,1256}. Each two blocks of Φ_1 have a 2-intersection.

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3 blocks: C = {1234,1256,1278} and D = {1234,1256,1357}.

The three blocks have 1 as a common element. They may have one more common element, but need not.

Theorem 4 provides that there are just two non-isomorphic subfamilies of Φ_1 having four blocks and just one for each number of blocks between five and seven. We prefer to choose lexicographically the first representatives than to take the complementary subfamilies.

4 blocks: E = {1234,1256,1357,1467} and
F = {1234,1256,1278,1357}

The common intersection of the three missing 4-sets of Φ_1 may be of cardinality 2 or 1

5 blocks: G = {1234,1256,1278,1357,1368}

6 blocks: H = {1234,1256,1278,1357,1368,1458}

7 blocks: Φ_1.

NON-ISOMORPHIC SUBFAMILIES OF Φ WHICH HAVE AT MOST SEVEN BLOCKS

0 blocks: $\emptyset$

1 block: A

2 blocks: B, AU{5678}. The two blocks may be disjoint.

3 blocks: C, D, BU{3456}, BU{3478}. None of the blocks containing 2 may be added to B, otherwise 2 would appear more frequently than 1. For a similar reason it is impossible to choose the subfamily A from Φ_1. The block from $\Phi\backslash\Phi_1$ may either intersect both blocks of B or just one of them.

4 blocks: E, F, CU{3456}, DU{2358}, DU{2367}
DU{2468}, BU{3478,5678}. None of the blocks with 2 may be added to C. Each of the remaining blocks (in $\Phi\backslash\Phi_1$) is the complement of a block of C. Similarly none of the blocks containing 2, or 3456, may be added to B. Each of 2367,2457, 3456, constitutes with D up to an isomorphism unique subfamily of four heptahedron hyperplanes, without an element common for

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all the four (i.e.,which is not isomorphic to E). The set 2358 is the only one from $\Phi \setminus \Phi_1$, which satisfies the following two conditions: it is not included in the ground-set of D and has a non-empty intersection with all sets of D. All complements (of sets) of D share the isomorphic position with respect to D.

5 blocks: G, E$\cup${2358}, E$\cup${2367}, F$\cup${2358}, F$\cup${2468}, F$\cup${3456}, C$\cup${3456,3478}, D$\cup${2468,3456}, D$\cup${2468,3478}.

The elements 2,3,4,5,6,7 share the isomorphic position in E and the same conclusion holds for the sets of E. Adding one of the complements to E we obtain therefore four isomorphic families, while by the adding of one of 2367,2457, 3456 we obtain the (up to an isomorphism unique) family of five heptahedron hyperplanes. Note that the set 1357 has a special position in F (with respect to the other three blocks). The additions of 2468 and one of the remaining three complements to F give rise consequently to two non-isomorphic families. The blocks 2358,2367 and 2457 share the isomorphic position with respect to F. As for C, two of the blocks 3456, 3478,5678 must be chosen, which gives three isomorphic possibilities. When D is considered, notice that none of the elements 2,3,5 may appear twice in the blocks of $\Phi \setminus \Phi_1$. We must therefore choose at least one block from each of the families {3456,3478,5678}, {2457,2468,5678} and {2367,2468, 3478}. Obviously one of the "intersection blocks" 2468,3478 and 5678 must be chosen, but the third family may be represented by the block, which does not exist in the other two. Since (5·4):8>2, some elements of S must occur thrice in five blocks and the subfamily B of Φ_1 must not be chosen.

6 blocks: H, G$\cup${2358}, G$\cup${2457}, G$\cup${5678}, E$\cup${2367,2457}, E$\cup${2358,2468}, F$\cup${2358,3456}, F$\cup${2367,3456}, F$\cup${2468,3456}, F$\cup${3456,3478}, C$\cup${3456,3478,5678}, D$\cup${2468,3478,5678}

$\Phi_1 \setminus G$ = {1458,1467}, so the element 4 has a special position in G (besides 1). This implies that there are three

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non-isomorphic possibilities to add the sixth block to G; it either is the complement (of a block of G) or is not; in the first case we must make a difference between 5678 and the complements containing 4. There is, up to an isomorphism, just one family of six blocks with the ground-set of cardinality 7. It arises from E. The elements 2,3,4,5,6,7 share the isomorphic position in E and we still have two non-isomorphic possibilities; to add one or two complements to E. We ignore the first one, for it necessarily includes a 2-subset of S, which does not contain 1 and which appears in three blocks (such a situation would contradict our favorizing 1) and is isomorphic to the case FU{2367,3456}. The family F contains 1 four times and 2 three times; one of the added blocks must be therefore any of 3456,3478,5678. Notice that the elements 4,6,8 share the isomorphic position in F and so do 3,5,7. If the sixth block is not a complement of F, then there are two possibilities; its intersection with the only complement may be included in {3,5,7} or not. If both the blocks added to F are the complements of F, then we have two non-isomorphic cases depending on whether the "special" 2468 is included or not.

None of the blocks with 2 may be added to C; this leaves just one possibility. As 1 appears in three blocks of D, we have (because of (6·4):8 =3) that each element of S must occur in three blocks of the corresponding family of six blocks. Thus three blocks containing 8 should be added to D, but it is easy to see that 2358 must not be added.

7 blocks: Φ_1, HU{2358}, HU{2367}, GU{2358,2367} GU{2358,2457}, GU{2358,5678}, GU{2457,2468}, GU{2457,3456}, GU{2457,5678}, EU{2367,2457,3456}, EU{2358,2468,3478}, FU{2468,3456,3478}, FU{3456,3478,5678}, FU{2358,3478,5678}.

There are two non-isomorphic possibilities to add the seventh block to H; it may be either 2358 or one of the complements. We have already noticed that the element 4 plays a special role in G. The block 5678 is the only

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complement of G without 4. We may have none, one or two complements among the two blocks added to G, but in the last two cases the possibilities which include 5678 are not isomorphic to those which do not. What is more, 2 and 3 are the only elements of S which appear in exactly three blocks of G. On account of that fact, in the case when two complements with 4 are chosen, we make a difference between the case when the same one and the different two of the elements 2,3 appear in these two complements. Since the element 1 appears only four times in E, (when adding three blocks to E) at least one block must be chosen from each of the families {3456,3478,5678},{2457,2468,5678},{2358,2367,5678},{2367,2468,3478},{2358,2457,3478},{2358,2468,3456}. If the chosen three sets do not contain 8, then we have the family of heptahedron hyperplanes. The element 1 does not occur in the same 2-subset in three blocks of E. Any block from {2367,2457,3456} has a 2-intersection contained in a block of E, with any block from {2358,2468,3478,5678}. Thus the only possibility left is to add to E three of the four blocks containing 8.

As for the family F, it is obvious that at most one of the blocks containing 2 can be added. If all the three blocks added to F are the complements, then we differentiate the cases when the "special" 2468 is among them and when is not. If just two of the added blocks are the complements (none of them may be 2468), then there are two non-isomorphic cases depending on whether there exist two added hyperplanes with the intersection included in the "special" set {3,5,7} or not. The first of these cases, however, is isomorphic to the case GU{2457,5678}.

None of the subfamilies C,D of Φ_1 may be possibly used for the production of subfamilies of Φ with seven blocks. The average number of appearances of elements of S in seven blocks is 28:8>3, so there exists an element which appears in more than three blocks.

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We conclude that there are

$$2 \cdot (1 + 1 + 2 + 4 + 7 + 9 + 12) + 14 = 86$$

non-isomorphic B-families.

There is a very simple routine to check that all the constructed B-families are really pairwise non-isomorphic. In fact, each two of the constructed subfamilies of ϕ (with the same number of blocks) differ in at least one of the following:

a)) the deck of incidence numbers of elements 1,2,...,8, without regard to order

b)) the number of 2-subsets of S appearing in three blocks.

Since each element (respectively 2-subset) of S appears in exactly seven (respectively three) blocks of Φ, these differences are preserved with the complementary subfamilies. This immediately gives, without the use of Theorem 3, the number of non-isomorphic subfamilies of Φ with more than seven blocks, which is the same to the number of those with less then seven blocks. We cannot prove in this way, however, that some new non-isomorphic possibilities do not arise with larger numbers of blocks or, equivalently, that a)) and b)) completely determine (up to an isomorphism) a subfamily of ϕ.

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REZIME

NOVA KONSTRUKCIJA PEJVING MATROIDA RANGA 4 NA SKUPU OD 8 ELEMENATA (I)

U radu |2| je pomoću kompjutera konstruisano svih 950 neizomorfnih prostih matroida na skupu od 8 elemenata.

Koristeći elementarne metode, bez pomoći kompjutera, mi izvodimo novu konstrukciju potklase P od 322 pejving matroida ranga 4 na skupu od 8 elemenata. (Pot)klasa P je po svojoj prilici najkomplikovanija, kad je u pitanju (ne)izomorfnost matroida, kod matroida na skupovima od najviše 8 elemenata.

Klasa P se razbija u tri disjunktne potklase. Konstrukcija prve dve je data u ovom radu, a konstrukcija treće potklase od P je data u narednom radu |1| (ova dva rada čine celinu).

Prilikom konstrukcije detaljno ispitujemo mogućnosti za javljanje neizomorfnih matroida. Pritom se u značajnoj meri služimo sa tri pomoćne klase grafova (koje su u obostrano jednoznačnoj korespodenciji sa odredjenim potklasama od P), kao i nekim osobinama Štajnerovog sistema S(3,4,8).

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ANOTHER CONSTRUCTION OF RANK 4 PAVING MATROIDS ON 8 ELEMENTS (II)

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ABSTRACT

In this paper, which is a sequel to |1|, we give the construction of all non-isomorphic C-families. This completes the construction of all non-isomorphic rank 4 paving matroids on 8 elements.

PRELIMINARIES

The reader should primarily read the preliminaries and the introduction to paper |1|; these will be used without reference.

A C-family is a family F of distinct 4-subsets of $S = \{1,2,3,4,5,6,7,8\}$ satisfying:

(a) $(X_1, X_2 \in F \wedge X_1 \neq X_2) \Rightarrow |X_1 \cap X_2| \leq 2$

(b) $(\exists X_1)\ (\exists X_2)\ (X_1 \in F \wedge X_2 \in F \wedge |X_1 \cap X_2| = 1)$

A D-family is a C-family F in which the condition (b) can be replaced by the stronger

(c) $(\forall X_1)\ (X_1 \in F \Rightarrow (\exists X_2)\ (X_2 \in F \wedge |X_1 \cap X_2| = 1))$

We shall give the construction of all 184 non-isomorphic C-families. We shall primarily construct 36 non-isomoprhic D-families, after which the construction of the remaining non-isomorphic C-families is considerably simplified.

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FURTHER DEFINITIONS

A C-family F arises from a D-family F´ if F´ is the (unique) maximal D-subfamily of F.

A 4-set X is permitted for a C-family F, which does not contain X, if F∪{X} is a C-family.

A 4-set X is 2-permitted for a D-family F, which does not contain X, if X is permitted for F, but F∪{X} is NOT a D-family.

A D-graph of a D-family F is a graph G such that

(i) the vertices of G are 4-sets of F

(ii) there is an edge {X,Y} in G if and only if $|X \cap Y| = 1$

We define a binary relation, denoted by "→", on the set of edges of a D-graph G, as follows:

If $a = \{A_1, A_2\}$ and $b = \{B_1, B_2\}$ are two edges of G, then

$$a \to b \overset{\text{def}}{\Longleftrightarrow} A_1 \cap A_2 = S \setminus (B_1 \cup B_2)$$

A marked D-graph is a graph G with a given binary relation ϕ on its edges, such that there exists a D-family with the D-graph G_1, which satisfies the following condition:

"There is an isomorphism of G onto G_1, which maps ϕ to →". The relation ϕ determines a markation of G.

Let F be a C-family, which satisfies the following condition: "The D-graph of the (unique) maximal D-subfamily of F is a star". Let the central vertex of the star (if the star is just an edge, then an arbitrary vertex) be the 4-set denoted as {a,b,c,d}.

We define the C-graph G of F as follows:

(i) the vertices of G are a,b,c,d

(ii) there is an edge {x,y} in G if and only if there is a 4-set in F, which contains {x,y} and which is different from {a,b,c,d}.

We define a 2-colouring of the defined C-graph G as follows: A vertex x of G is black if and only if there exists a 4-set X in F such that $X \cap \{a,b,c,d\} = \{x\}$. The vertex x is white otherwise.

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A coloured C-graph is a graph G on four vertices, each of which is coloured either white or black, satisfying the condition that there exists at least one black vertex. The colours of the vertices determine a colouring of G.

In is easy to prove that the coloured C-graphs are actually C-graphs of some C-families.

CONSTRUCTION OF NON-ISOMORPHIC D-FAMILIES

We begin with a number of easy lemmas describing some properties of D-graphs and marked D-graphs. Some of them will be used in our construction of D-families. They may offer a better insight into the D-graphs which we construct and may be used for another approach to the construction as well. These lemmas are denoted by apostrophes, for they do not relate to any particular theorem by this time.

LEMMA 1´. *Each element of S corresponds to at most one edge of a D-graph, in the sense that it is the intersection of the two end-vertices of that edge.*

REMARK: In the figures of D-graphs the edges will be denoted by the corresponding intersection elements.

P r o o f. Suppose that the same element of S corresponds to two edges of a D-graph. It is easy to check that at least 9 different elements are needed for their vertices, regardless of whether these two edges have a common vertex or not.

LEMMA 2´. *A D-graph cannot have more than 8 edges.*

P r o o f. Immediate consequence of Lemma 1´.

LEMMA 3´. *The maximal degree of a vertex of a D-graph is 4.*

P r o o f. Immediate consequence of Lemma 1´.

LEMMA 4´. *Each element of S corresponds to at most one edge of a D-graph, in the sense that it is the complement (with respect to S) of the union of the two end-vertices of that edge.*

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P r o o f. Seven elements are not sufficient for the end-vertices of two edges in any case.

LEMMA 5′. *For each edge x of a D-graph there exists at most one edge y such that $x \to y$ and at most one edge z such that $z \to x$.*

P r o o f. The two statements follow immediately from Lemmas 4 and 1 respectively.

LEMMA 6′. *If $A_1A_2A_3A_4$ is a 3-path of a D-graph, then it holds that at least one of $A_1A_2 \to A_3A_4$ and $A_3A_4 \to A_1A_2$ is true.*

P r o o f. Suppose that $A_1A_2 \to A_3A_4$ is not true, that is, $S \setminus (A_3 \cup A_4) \neq A_1 \cap A_2$. $A_1 \cap A_2 \not\subset A_3$ by Lemma 1′ and so we have $A_1 \cap A_2 \subset A_4$. This implies that $S \setminus (A_1 \cup A_2) \subset A_4$ (otherwise A_4 would have a 3-intersection with some of A_1, A_2). Since $|A_3 \cap A_2| = 1$ and $|A_3 \cap A_1| \leq 2$, we have also $S \setminus (A_1 \cup A_2) \subset A_3$, which gives $A_3A_4 \to A_1A_2$

LEMMA 7′. *No two vertices of a D-graph may have the empty intersection.*

P r o o f. Otherwise one of them would have a 3-intersection with any vertex incident to the other.

LEMMA 8′. *If K and L are two incident vertices of a D-graph and $K \cap L = \{p\}$, $S \setminus (K \cup L) = \{q\}$, then $L = \{p\} + ((S \setminus K) \cup \{q\})$*

P r o o f. Trivial.

LEMMA 9′. *If $x = A_1A_2$ and $y = B_1B_2$ are two edges of a D-graph such that $x \to y$, then there exists an edge adjacent to both x and y.*

P r o o f. $B_1 \cap B_2$ does not belong to both A_1 and A_2 by Lemma 1′. Suppose that $B_1 \cap B_2 \neq A_1$ (similarly if $B_1 \cap B_2 \neq A_2$). $B_1 \cup B_2 \neq A_1 \cap A_2$ by the assumption. Lemma 7′ combined with $|A_1 \setminus A_2| = 1$ gives that either $|A_1 \cap B_1| = 1$ or $|A_1 \cap B_2| = 1$, i.e., there exists one of the edges A_1B_1 and A_1B_2.

LEMMA 10′. *If a D-graph contains the edges A_1A_2 and B_1B_2 such that $A_1A_2 \to B_1B_2$, and does not contain any of the edges A_1B_1 and A_1B_2, then $A_1 \supset B_1 \cap B_2$.*

P r o o f. Immediately follows from the previous proof.

LEMMA 11′. *There are no odd cycles in any D-graph.*

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P r o o f. We have to exclude just the 3-, 5- and 7-cycles, by Lemma 2´.3-cycles: A 3-cycle $A_1A_2A_3$ of a D-graph yields

$A_3 \cap (A_1 \cap A_2) = \emptyset$ and $|A_3 \cap (A_1 \setminus A_2)| = |A_3 \cap (A_2 \setminus A_1)| = 1$,

which implies $|A_3| \leqslant 3$, a contradiction.

5-cycles: Suppose that there exists a 5-cycle $A_1A_2A_3A_4A_5$ (with edges A_iA_{i+1}, $1 \leqslant i \leqslant 5$, $A_{5+1} = A_1$) in a D-graph. We may assume by Lemmas 6´and 5´, without a loss of generality, that $A_1A_2 \to A_3A_4$ and $A_4A_5 \to A_1A_2$. The first assumption gives $A_4 \neq A_1 \cap A_2$, while the second, combined with Lemma 10´ and the non-existence of edges A_1A_4 and A_2A_4 gives $A_4 \supset A_1 \cap A_2$, a contradiction.

7-cycles: Let $A_1A_2A_3A_4A_5A_6A_7$ (edges A_iA_{i+1}, $1 \leqslant i \leqslant 7$, $A_{7+1} = A_1$) be a 7-cycle in a D-graph. Similarly as in the previous case, assume that $A_1A_2 \to A_3A_4$ and $A_6A_7 \to A_1A_2$. Then $A_4 \neq A_1 \cap A_2$, while Lemma 10´ gives $A_6 \supset A_1 \cap A_2$. Now $A_5 \supset A_1 \cap A_2$ contradicts Lemma 1´, while $A_5 \neq A_1 \cap A_2$ contradicts Lemma 4´.

LEMMA 12´. *If $A_1A_2A_3A_4$ is a 3-path of a D-graph G and both $A_1A_2 \to A_3A_4$ and $A_3A_4 \to A_1A_2$ are satisfied, then G contains the edge A_1A_4 and both $A_1A_4 \to A_2A_3$ and $A_2A_3 \to A_1A_4$ also hold.*

P r o o f. We have $a = A_1 \cap A_2 = S \setminus (A_3 \cup A_4)$ and $b = A_3 \cap A_4 = S \setminus (A_1 \cup A_2)$ by the assumption. Lemmas 7´ and 11´ provide that $|A_1 \cap A_3| = |A_2 \cap A_4)| = 2$. As $a \notin A_3 \cup A_4$ and $b \notin A_1 \cup A_2$, we have that none of the sets $A_1 \cap A_3$, $A_2 \cap A_4$ contains a or b. The assumption combined with Lemma 1´ gives that the sets $A_1 \cap A_3$, $A_2 \cap A_4$, $A_2 \cap A_3$ and $A_1 \cap A_4$ are mutually disjoint. It follows that

$A_1 \cap A_4 = S \setminus (\{a,b\} + (A_1 \cap A_3) + (A_2 \cap A_4) + (A_2 \cap A_3))$, which gives $|A_1 \cap A_4| = 1$. Since the sum in the brackets coincides with $A_2 \cup A_3$, we have $A_1A_4 \to A_2A_3$. The relation $A_2A_3 \to A_1A_4$ follows from $A_1 \cup A_4 = (A_1 \cap A_4) + \{a,b\} + (A_1 \cap A_3) + (A_2 \cap A_4)$

Our construction of non-isomorphic D-families will be somewhat similar to that of B-families(|1|). We shall start with examples of "small" D-families and in each case look for the non-isomorphic possibilities for the addition of the permitted, but not 2-permitted 4-sets (that is, those 4-sets, the addition of which gives a new D-family). We must, however,

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also take care about 2-permitted 4-sets for a D-family, because they may be vertices of a new connected component of an augmented D-graph. These 4-sets necessarily appear in the complementary pairs, which are denoted by a short line between their two 4-sets.

The main novelty in the construction of non-isomorphic D-families will be in that we shall mostly look for non-isomorphic possibilities for augmentation of the corresponding marked D-graphs, instead of the D-families themselves. Such an approach will be justified by the later Theorem 2.

The constructed non-isomorphic D-families will be given by means of their marked D-graphs in the table which follows the construction. The types of marked D-graphs given in the table will be denoted by the numbers in brackets. (We shall mention all these numbers during the construction). The same denotation will be often used for the types of non-isomorphic D-families, which correspond to the examples of the marked D-graphs given in the table.

CONSTRUCTION OF ALL THE NON-ISOMOPRHIC MARKED D-GRAPHS AND ONE MORE

We start with an example of the simplest D-graph,

$$1234 \overset{1}{\text{———}} 1567$$

We may assume, without any loss of generality, that each example (=representative), of the non-isomorphic marked D-graphs that we construct, contains this edge.

There are (with respect to this edge) 36 permitted 4-sets: 9 contain $\{1,8\}$, 9 are contained in $\{2,3,4,5,6,7\}$, 9 have 1-intersections with $\{1,2,3,4\}$ and 9 with $\{1,5,6,7\}$.

REMARK: An alternative proof that D-graphs have not 3-cycles follows from the fact that none of the permitted 4-sets has 1-intersections with both of 1234 and 1567.

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Adding any of the last 18 permitted 4-sets we obtain a D-graph of type (2). Adding any two 4-sets, such that they have exactly one element in common, we obtain a D-graph of type (3).

The following theorem refers to the remaining (types of) D-graphs:

THEOREM 1. *Each D-graph with at least three edges belongs to exactly one of the following four classes:*

a) graphs with at least one 4-cycle

b) graphs with at least one vertex of degree ≥ 3

c) graphs with at least one 3-path , which do not belong to any of the classes a) and b)

d) disconnected graphs without 3-paths

P r o o f. Will be implicit in what follows.

We give separate constructions for each of these four classes of D-graphs: (all the here mentioned graphs are D-graphs, even if it is not explicitly stated)

Class a) A 4-cycle contains a subgraph of type (2). Starting with the family F_o = {1234,1567,2568}, it is easy to check that the only 4-set, which gives rise to a 4-cycle (type (4)),is 3478. The only permitted 4-sets for the family F_1 = {1234, 1567,2568,3478} are 1278,1358,1368,1458,1468 and their complements. None of them has a 1-intersection with any set of F_1,i.e., a 4-cycle in a D-graph is always a connected component. However, some pairs of the permitted 4-sets have 1-intersections. We choose 1358,2457 and so obtain (a D-graph of) type (5) with the permitted 4-sets just 1278-3456,1468, 2367. Adding further one (respectively both) of 1468,2367 we obtain type (6) (respectively type (7)).

Class b) Adding any of the 4-sets 3578,3678,4578,4678 to the family F_o = {1234,1567,2568}, we obtain a star (type (8)), with the centre 1234. Suppose that 3578 is chosen. The permitted 4-sets of the family F = {1234,1567,2568,3578} are 4678,1468,1478,2467,2478,3467,3468,1278-3456,1368-2457,1458-2367. As there are no 1-intersections among the last six permitted (= 2-permitted) 4-sets, we conclude that all the graphs in class b) are connected.

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The only possibility to obtain a star with four legs (type (9)) is to add 4678 to F. It is a maximal D-graph, for none of its permitted 4-sets (the last six of the list above) has 1-intersections with any of 1234,1567,2568,3578,4678.

Adding any of 1468,1478,2467,2478,3467,3468 to F, we obtain a D-graph of type (10). If 1468 is chosen, then the remaining permitted 4-sets are 2367,2457,2467,2478,3467,1278-3456. If we add 2467 to the family G = {1234,1567,2568,3578, 1468}, then we obtain a D-graph of type (11), which is easily seen to be a maximal one. If we add 2478 or 3467 to G, then we obtain two isomorphic (marked) D-graphs of type (12). However, if we add 2367, respectively 2457, to G, then we obtain two isomoprhic graphs, which have a different markation (types (13) and (14)).

In the next step we explore the possibilities for the augmentation of D-graphs corresponding to the families $G_1 = G \cup \{2478\}$, $G_2 = G \cup \{2367\}$, $G_3 = G \cup \{2457\}$, with the families of the permitted 4-sets {2367,3456,3467}, {2457,2478,1278-3456} and {2367,3467,1278-3456} respectively.

Adding 3467 to G_1 we obtain a maximal D-graph of type (15). Adding 2457 to G_2 (identically, 2367 to G_3), we obtain a maximal D-graph of type (16). Adding 2367 to G_1 or 2478 to G_2, we obtain two isomorphic marked D-graphs of type (17). Adding 3456 to G_1 or 3467 to G_3, we obtain two isomorphic marked D-graphs of type (18) (isomorphic to the graphs of type (17), but with a different markation). Finally, adding (the only possible) 3456 to $G_1 \cup \{2367\}$ (identically, 2367 to $G_1 \cup \{3456\}$), we obtain a maximal D-graph of type (19).

Class c) We start for the third time from the family $\bar{F}_o = \{\bar{1}\bar{2}\bar{3}\bar{4},\bar{1}\bar{5}\bar{6}\bar{7},\bar{2}\bar{5}\bar{6}\bar{8}\}$. This time we add some of the remaining permitted 4-sets having 1-intersections with 1567 or 2568, i.e. some of 1378,1478,2378,2478,3457,3458,3467,3468 and obtain a D-graph of type (20). If we choose the family $H = \{\bar{1}\bar{2}\bar{3}\bar{4},\bar{1}\bar{5}\bar{6}\bar{7},\bar{2}\bar{5}\bar{6}\bar{8},\bar{1}\bar{3}\bar{7}\bar{8}\}$, then the permitted 4-sets are 2457,2467,2478,3456,3457,3458,3467,3468,4578,4678,1458-2367, 1468-2357. The addition of any of 3457,3467,4578,4678 to H

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gives rise to a D-graph of type (10). The addition of 2478, respectively 3456, (to H) yields marked D-graphs of types (21), respectively (22), while the addition of any of 2457,2467,3458, 3468 gives a 4-path marked in a third style (type (23)). We have here also one (up to an isomoprhism) possibility to obtain an edge of another component of a D-graph, by the addition of both of, for example, 1458,2357 to H. Thus we obtain the type (24).

We denote the families HU{2478}, HU{3456}, HU{2457} by H_1, H_2, H_3 respectively. Their families of permitted 4-sets are

{3456,3457,3458,3467,3468,1458-2367,1468-2357},
{2457,2467,2478,4578,4678,1458-2367,1468-2357} and
{1468,3456,3458,3467,3468,4678,1458-2367} respectively.

We omit all those possibilities for the augmentation of the D-graphs of types (21)-(23), in which a vertex of degree 3 is produced.

If we add 3456 to H_1 (identically, 2478 to H_2), then we obtain the marked D-graph of type (25). Note, however, that we can here also obtain three types of disconnected marked D-graphs (types (26), (27) and (28)) by adding, for example, 1458 and 2357 to H_1, H_2 and $H_1 \cup \{3456\}$ respectively. If we add 3468 to H_3, then we obtain a maximal marked D-graph of type (29), which differs only by markation from the D-graph of type (25). If we add 1468 or 3458 to H_3, then we obtain a D-graph of type (30). The only permitted 4-sets for the family $H_3 \cup \{1468\}$ are 2367,3458,3456,3467. We neglect the last two, because they yield a vertex of degree 3. If any of 2367,3458 is added to $H_3 \cup \{1468\}$, then we obtain a D-graph of type (31); if both are added, then we have a maximal D-graph of type (32). Class d). If we start again with F_o, then we should add only some of those permitted 4-sets, which have no 1-intersections with any of the first three, that is, some of

1278-3456, 1358-2467, 1368-2457, 1458-2367, 1468-2357

Note that 1278 and 3456 have not 1-intersections with any of these 4-sets, so they cannot be the vertices of a

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D-graph. Adding any of {1358,2367}, {1368,2357}, {1458,2467}, {1468,2457}, respectively any of {1358,2457},{1368,2467}, {1458,2357}, {1468,2367}, to the family F_o, we obtain D-graphs of types (33), respectively (34). The D-graphs of these two types are isomorphic and without markation, but observe the following difference between them: The intersection element, which corresponds to the one-edge component, is contained in the vertex of degree 2 with the type (33), which is not the case with the type (34).

The only permitted 4-sets, apart from 1278-3456, for $F_o \cup \{1358,2367\}$, respectively $F_o \cup \{1358,2457\}$, which do not augment the component of F_o, are 1468,2457, respectively 1468,2367. By adding any, but just one, of these 4-sets, we obtain a D-graph of type (35), while the addition of both permitted 4-sets in both cases yields a 4-cycle (type (6)).

The only remaining case is when each component of a D-graph has just one edge. We start with the D-family $J = \{1234,1567,1258,2367\}$ (corresponding to the D-graph of type (3)). The only permitted 4-sets, which do not give rise to a component with (at least) two edges, are 1368-2457 and 1378-2456. We obtain a D-graph of type (36) by the adding of any of {1368,2456}, {1378,2457} to J.

We give (on the next two pages) the table of (examples of) D-graphs corresponding to non-isomorphic D-families. This last notion coincides with "non-isomorphic marked D-graphs",*) except for the types (33) and (34), where two isomorphic (marked) D-graphs correspond to non-isomorphic D-families.

Apart from the explained designations, we give, beside the number of the type, for each D-graph of the table, the number of non-isomorphic C-families having the maximal D-subfamily of the corresponding type. The production of these numbers will be explained in the last section.

The markation (the edges related by →) will be denoted only in the cases when it is necessary for distinguishing two isomorphic graphs corresponding to two non-isomorphic D-families

*) Two marked D-graphs G_1 and G_2 are isomorphic if there is a graph-isomorp which maps G_1 onto G_2 and preserves the markation.

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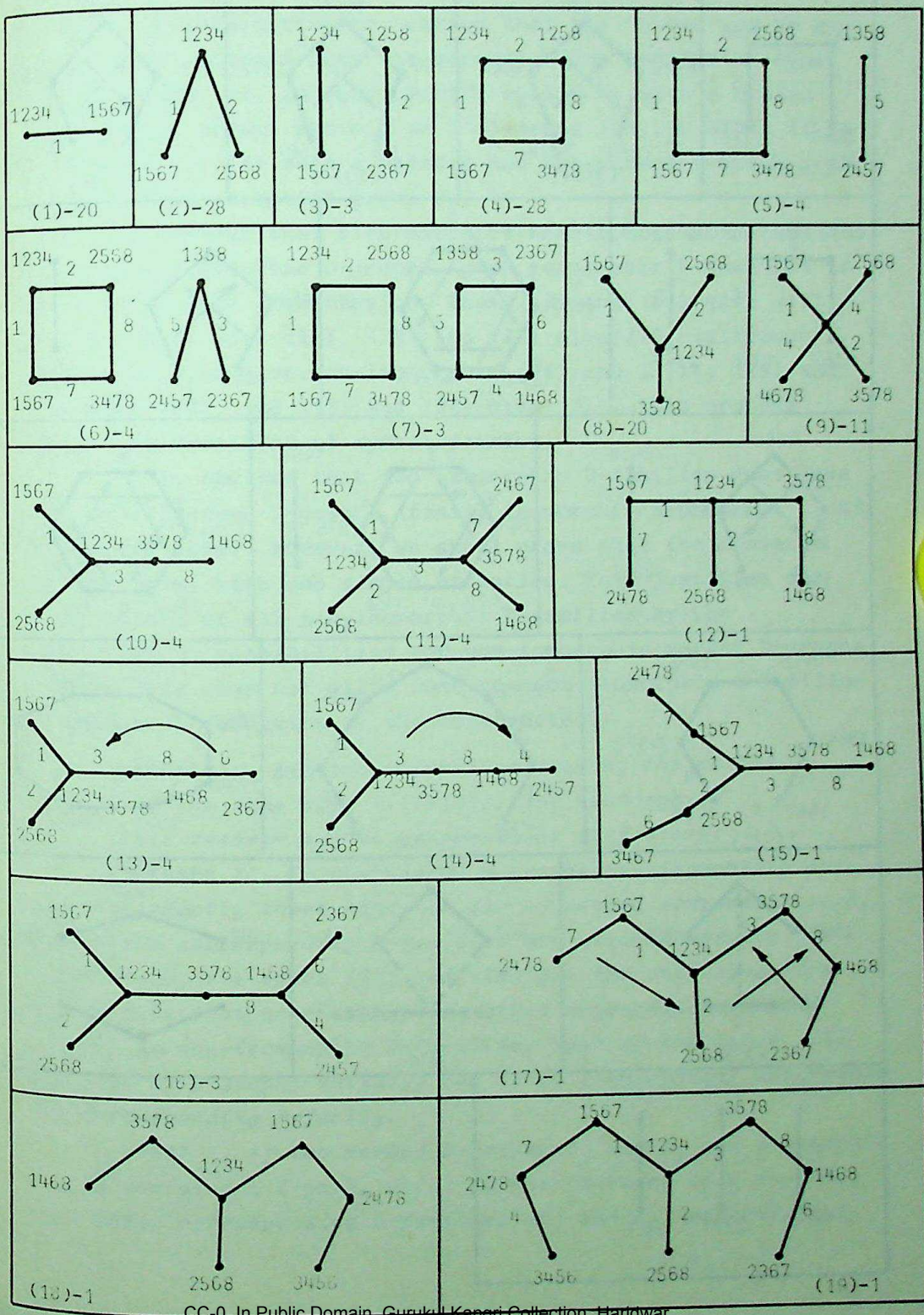
THE TABLE OF (EXAMPLES OF) D-GRAPHS CORRESPONDING TO NON-ISOMORPHIC D-FAMILIES



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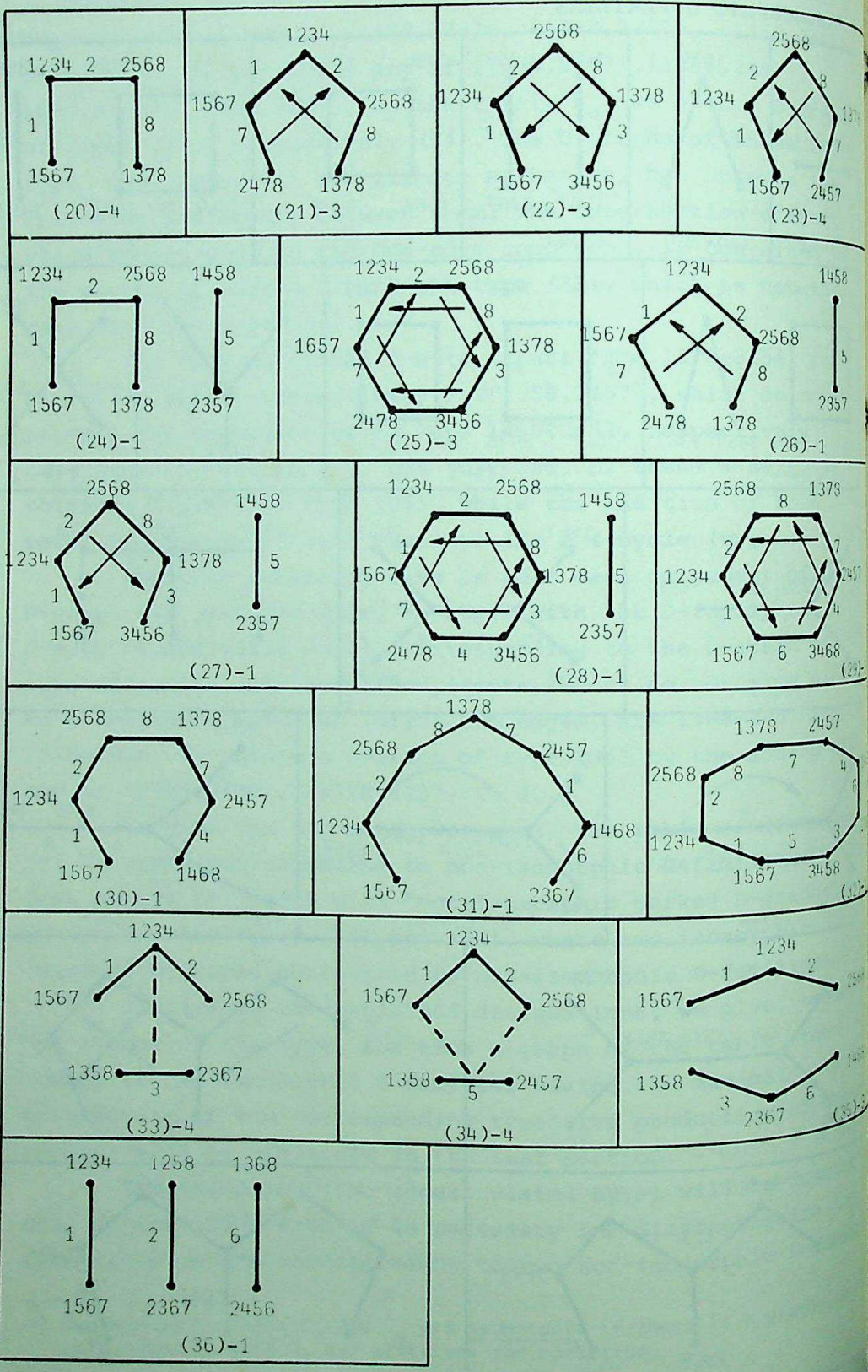
THE TABLE OF D-GRAPHS (CONTINUED)



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It might be interesting to mention that the unique (up to an isomorphism) matroid with the corresponding D-graph of type (32) (8-cycle) is, in fact, Piff's cyclic (= with a cyclic automorphism group) matroid on 8 elements ([3],p. 330). It is quite natural that such a matroid has the marked D-graph with the cyclic automorphism group C_8.

We observe that five pairs of non-isomoprhic D-families, which have isomoprhic D-graphs, give isomorphic D-families in the next step of augmentation. Thus (13) and (14) give (16), (17) and (18) give (19), (22) and (23) give (25) (although they could also give non-isomorphic (26) and (27)), (26) and (27) give (28), and (33) and (34) give (35). This process looks like some kind of symmetrization.

It is obvious that two isomorphic D-families must have isomorphic marked D-graphs (family isomorphisms preserve 1-intersections). However, we shall prove that the converse is also true, with one single exception. This justifies our construction of all non-isomorphic D-families by the exhaustion of possibilities for non-isomorphic marked D-graphs, because this does not allow some new non-isomorphic D-families to appear, in addition to the constructed.

THEOREM 2. *Any two D-families (having the marked D-graphs) of the same type ((1)-(36)) are isomorphic.*

This theorem may be reformulated as follows:

THEOREM 2'. *If two marked D-graphs are isomorphic and have not exactly three edges in two connected components each, then their corresponding D-families are also isomorphic. In other words, the types (33) and (34) are the only exception to the rule that non-isomorphic marked D-graphs correspond exactly to non-isomorphic D-families. Each of the types (33) and (34) of D-graphs uniquely (up to an isomorphism) determines the corresponding D-family.*

LEMMA 1. *If two marked D-graphs G_1 and G_2 are isomorphic and G_1 contains a 3-path, which is not included in a 4-cycle, then their corresponding D-families (F_1 and F_2 respectively)*

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are also isomorphic.

P r o o f of Lemma 1. Let $\gamma: G_1 \to G_2$ be the given isomorphism and let $X_1Y_1Z_1T_1$ be a 3-path in G_1, such that X_1T_1 is not an edge of G_1. The image of $X_1Y_1Z_1T_1$ under γ (in G_2) is denoted by $X_2Y_2Z_2T_2$. Lemmas 6′ and 12′ give that either $X_1Y_1 \to Z_1T_1$ or $Z_1T_1 \to X_1Y_1$. We suppose the first, the other case is treated quite similarly. We have $X_2Y_2 \to Z_2T_2$ and denote for $i \in \{1,2\}$:

$X_i \cap Y_i = a_i$, $Y_i \cap Z_i = b_i$, $Z_i \cap T_i = c_i$, $S \setminus (X_i \cup Y_i) = d_i$,
$S \setminus (Y_i \cup Z_i) = e_i$, $(X_i \cap Z_i) \setminus T_i = f_i$, $Y_i \cap T_i = \{g_i, h_i\}$

We claim that at least one of the permutations

$$\alpha = \begin{pmatrix} a_1 b_1 c_1 d_1 e_1 f_1 g_1 h_1 \\ a_2 b_2 c_2 d_2 e_2 f_2 g_2 h_2 \end{pmatrix} \qquad \beta = \begin{pmatrix} a_1 b_1 c_1 d_1 e_1 f_1 g_1 h_1 \\ a_2 b_2 c_2 d_2 e_2 f_2 h_2 g_2 \end{pmatrix}$$

induces an isomorphism of F_1 onto F_2, which would complete the proof.

For $i \in \{1,2\}$ we have:

$X_i = \{a_i, c_i, e_i, f_i\}$; $Y_i = \{a_i, b_i, g_i, h_i\}$; $Z_i = \{b_i, c_i, d_i, f_i,\}$;
$T_i = \{c_i, e_i, g_i, h_i\}$

Namely, X_i obviously contains a_i and f_i, X_i contains e_i by Lemma 4′, and contains c_i, for we have not $Z_iT_i \to X_iY_i$. The other three pairs of sets are easily deduced by applying Lemma 8′.

Thus both α and β map the sets X_1, Y_1, Z_1, T_1 to the sets X_2, Y_2, Z_2, T_2 respectively.

The only 4-sets, which are permitted for X_i, Y_i, Z_i, T_i, are $(i=1,2)$:

$A_i^1 = \{a_i, c_i, d_i, g_i\}$; $A_i^3 = \{b_i, e_i, f_i, g_i\}$; $A_i^5 = \{a_i, b_i, d_i, e_i,\}$;
$A_i^6 = \{d_i, f_i, g_i, h_i\}$

$A_i^2 = \{a_i, c_i, d_i, h_i\}$; $A_i^4 = \{b_i, e_i, f_i . h_i\}$

$A_i^7 = \{a_i, d_i, f_i, g_i\}$; $A_i^9 = \{b_i, d_i, e_i, g_i\}$; $A^{11}_i = \{a_i, d_i, e_i, g_i\}$;
$A^{13}_i = \{d_i, e_i, f_i, g_i\}$

$A_i^8 = \{a_i, d_i, f_i, h_i\}$; $A^{10}_i = \{b_i, d_i, e_i, h_i\}$; $A^{12}_i = \{a_i, d_i, e_i, h_i\}$;
$A^{14}_i = \{d_i, e_i, f_i, h_i\}$

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Let W_1 belong to $\{A_1^1,\ldots,A_1^{14}\}\cap F_1$. We shall primarily prove that either $\alpha(W_1)\in F_2$ or $\beta(W_1)\in F_2$ or both (equivalently, that holds at least one of $\alpha(W_1)=\gamma(W_1)$, $\beta(W_1)=\gamma(W_1)$).

If $W_1\in\{A_1^{13}, A_1^{14}\}$, then there exists the edge Y_1W_1 in G_1, which implies the existence of the edge $Y_2\,\gamma(W_1)$ in G_2 and gives $\gamma(W_1)\in\{A_2^{13}, A_2^{14}\}$, providing that either $\gamma(W_1)=\alpha(W_1)$ or $\gamma(W_1)=\beta(W_1)$. We similarly derive, by keeping the incidence with Z_i,X_i,T_i respectively, that

$$W_1\in\{A_1^{11}, A_1^{12}\}\Rightarrow\gamma(W_1)\in\{A_2^{11}, A_2^{12}\}$$

$$W_1\in\{A_1^{6},A_i^{9},A_1^{10}\}\Longrightarrow\gamma(W_1)\in\{A_2^{6},A_2^{9},A_2^{10}\}$$

$$W_1\in\{A_1^{5},A_1^{7},A_1^{8}\}\Longrightarrow\gamma(W_1)\in\{A_2^{5},A_2^{7},A_2^{8}\}$$

As the inverse argument also holds, we have

$$W_1\in\{A_1^1,A_1^2,A_1^3,A_1^4\}\Longrightarrow\gamma(W_1)\in\{A_2^1,A_2^2,A_2^3,A_2^4\}$$

However, since γ preserves the markation, we cannot have, for example, $\gamma(A_1^6)=A_2^9$, otherwise the subgraph of G_1 with vertices A_1^6,X_1,Y_1,Z_1,T_1 would not have the same markation as its image under γ. By a similar argument we derive:

$$W_1=A_1^5\Rightarrow\gamma(W_1)=A_2^5\quad;\quad W_1=A_1^6\Rightarrow\gamma(W_1)=A_2^6$$

$$W_1\in\{A_1^7,A_1^8\}\Rightarrow\gamma(W_1)\in\{A_2^7,A_2^8\}$$

$$W_1\in\{A_1^9,A_1^{10}\}\Rightarrow\gamma(W_1)\in\{A_2^9,A_2^{10}\}$$

Thus $\gamma(W_1)$, which is a set if F_2, equals at least one of $\alpha(W_1)$, $\beta(W_1)$ for $W_1\in\{A_1^5,\ldots,A_1^{14}\}$.

If the set $W_1\in\{A_1^1,A_1^2,A_1^3,A_1^4\}$, then the vertex W_1 is not incident (in G_1) to any of the vertices X_1,Y_1,Z_1,T_1. Since G_1 does not contain isolated vertices, W_1 is joined to another vertex V_1 of G_1. It is easy to check that ordered pair (W_1,V_1) must be one of the following twelve

$(A_1^1,A_1^3),(A_1^3,A_1^1),(A_1^2,A_1^4),(A_1^4,A_1^2)$ ....., $(A_1^1,A_1^{10}),(A_1^1,A_1^{14})$.

$(A_1^2,A_1^9),(A_1^2,A_1^{13}),(A_1^3,A_1^8),(A_1^3,A_1^{12}),(A_1^4,A_1^7),(A_1^4,A_1^{11})$

while $\gamma(W_1,V_1)$ must be one of the ordered pairs obtained from the above pairs by replacing the lower indices 1 by 2.

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In the first four cases, we use that γ preserves the edges and maps $\{A_1^1,\ldots,A_1^4\}$ to $\{A_2^1,\ldots,A_2^4\}$. This gives, for example, that

either $\quad \gamma(\{A_1^1,A_1^3\}) = \{A_2^1,A_2^3\} = \alpha(\{A_1^1,A_1^3\})$

or $\quad \gamma(\{A_1^1,A_1^3\}) = \{A_2^2,A_2^4\} = \beta(\{A_1^1,A_1^3\})$

Since $\gamma(A_1^{10}) \in \{A_2^9,A_2^{10}\}$, we have that

either $\quad \gamma(\{A_1^1,A_1^{10}\}) = \{A_2^1,A_2^{10}\} = \alpha(\{A_1^1,A_1^{10}\})$

or $\quad \gamma(\{A_1^1,A_1^{10}\}) = \{A_2^2,A_2^9\} = \beta(\{A_1^1,A_1^{10}\})$

The remaining seven cases are treated similarly.

Now we know that each set of F_1 is mapped by at least one of α,β to a set of F_2. The only kind of counterexample to our claim, which could possibly arise, should be of the following form:

There exist two sets P_1,Q_1 in F_1 satisfying

$$\alpha(P_1) \in F_2 \ ; \ \beta(P_1) \notin F_2 \ ; \ \alpha(Q_1) \notin F_2 \ ; \ \beta(Q_1) \in F_2$$

Since P_1 and Q_1 have different images under α and β, it follows that

$$|P_1 \cap \{g_1,h_1\}| = |Q_1 \cap \{g_1,h_1\}| = 1$$

In any of the four possible cases we have

$$||P_1 \cap Q_1| - |\alpha(P_1) \cap \beta(Q_1)|| = 1$$

The sets $\alpha(P_1)$ and $\beta(Q_1)$ are vertices of G_2, which are images of P_1,Q_1 respectively under the D-graph isomorphism γ. So we must have

$$|P_1 \cap Q_1| = 1 \Leftrightarrow |\alpha(P_1) \cap \beta(Q_1)| = 1$$

Since both $|P_1 \cap Q_1|$ and $|\alpha(P_1) \cap \beta(Q_1)|$ belong to $\{0,1,2\}$, we have a contradiction, which proves Lemma 1.

P r o o f of Theorem 6. Due to Lemma 1, we just have to prove the uniqueness (up to an isomorphism) of D-families corresponding to D-graphs of the types (1)-(9) and (33)-(36) inclusive. We construct some of the possible isomorphisms between any two D-families (having D-graphs) of these types. When defining the corresponding permutations of S, we shall use some of the Lemmas 1´,4´,9´,12´ to prove that the required conditions are uncontradictory, but we shall not state this

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explicitly in most of the particular cases.

Stars (types (1), (2), (8), (9)): Given two D-graphs G_1 and G_2, which are stars of the same type, we denote their centres (arbitrarily chosen in case (1)) by X_1, X_2 respectively and other vertices of degree 1 by Y_1^i, Y_2^i respectively ($i \in N$; the upper bound for i varies from 1 to 4, depending on the type).

Any permutation α of S, which satisfies the conditions:

$$\alpha(X_1)=X_2 \; ; \quad \alpha(X_1 \cap Y_1^i)=X_2 \cap Y_2^i \; ; \quad \alpha(S \setminus (X_1 \cup Y_1^i))=S \setminus (X_2 \cup Y_2^i)$$

for each i

establishes an isomorphism between the D-families F_1, F_2 corresponding to G_1, G_2 respectively. Namely, α maps the sets of F_1 onto the sets of F_2, because all non-central vertices are uniquely determined by Lemma 8´.

Types (3) and (36): Lemmas 1´ and 9´ give that each of some two disconnected edges of a D-graph has exactly one vertex "containing the intersection element of the other edge"

Let X_1Y_1 and Z_1T_1 be two (in the case of (36), arbitrary two) disconnected edges of G_1 and let X_2Y_2 and Z_2T_2 be the corresponding two edges of G_2, so that the vertices denoted by X and Z have the property described in the previous sentence.

Any of the permutations α_1, α_2 of S satisfying (for i=1,2)

$$\alpha_i(X_1)=X_2 \; ; \quad \alpha_i(Z_1)=Z_2 \; ; \quad \alpha_i(X_1 \cap Y_1)=X_2 \cap Y_2 \; ; \quad \alpha_i(Z_1 \cap T_1)=Z_2 \cap T_2$$

$$\alpha_i(S \setminus (X_1 \cup Y_1))=S \setminus (S_2 \cup Y_2) \; ; \quad \alpha_i(S \setminus (Z_1 \cup T_1))=S \setminus (Z_2 \cup T_2)$$

maps $\{X_1, Y_1 Z_1, T_1\}$ to $\{X_2, Y_2, Z_2, T_2\}$, that is, it establishes an isomorphism between the corresponding D-families of type (3).

As for the case (36), we denote the remaining pair of edges in G_1 and G_2 by U_1V_1, respectively U_2V_2. It is easy to check that one of the permutations α_1, α_2 maps $\{U_1, V_1\}$ to $\{U_2, V_2\}$, that is, it establishes also an isomorphism between the corresponding D-families of tipe (36).

D-graphs with a 4-cycle (types (4), (5), (6), (7)):

Let $X_1Y_1Z_1T_1$ and $X_2Y_2Z_2T_2$ be the two (in the case of (7) arbitrary two) corresponding 4-cycles of the isomorphic D-graphs G_1 and G_2. Each of the permutations α of S, satisfying,

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for example:

$\alpha(X_1 \cap Y_1) = X_2 \cap Y_2$; $\alpha(Y_1 \cap Z_1) = Y_2 \cap Z_2$ (*)

$\alpha(Z_1 \cap T_1) = Z_2 \cap T_2$; $\alpha(T_1 \cap X_1) = T_2 \cap X_2$; $\alpha(X_1 \cap Z_1) = X_2 \cap Z_2$

maps $\{X_1, Y_1, Z_1, T_1\}$ to $\{X_2, Y_2, Z_2, T_2\}$

that is, it establishes an isomorphism between the corresponding D-families (in the case) of type (4).

As for the type (7), we denote the other two 4-cycles of G_1 and G_2 by $P_1Q_1R_1S_1$ and $P_2Q_2R_2S_2$ respectively. We require additionally

$$\alpha(P_1 \cap Q_1) = P_2 \cap Q_2 \quad ; \quad \alpha(Q_1 \cap R_1) = Q_2 \cap R_2$$

$$\alpha(R_1 \cap S_1) = R_2 \cap S_2 \quad ; \quad \alpha(S_1 \cap T_1) = S_2 \cap T_2$$

This is not a contradiction with the previous conditions on α. In particular, the following property is preserved with such α: 2-intersections of non-incident vertices of one 4-cycle exactly correspond to the (unordered) pairs of 1-intersections corresponding to the non-incident edges of the other.

The only permutation α, which satisfies all the above conditions, establishes an isomorphism between the corresponding D-families of type (7).

Type (5): If the remaining edges of G_1 and G_2 are denoted by U_1V_1 and U_2V_2 respectively, then we require, in addition to (*)

$\alpha(U_1 \cap V_1) = U_2 \cap V_2$ $\quad \alpha(S \setminus (U_1 \cup V_1)) = S \setminus (U_2 \cup V_2)$ (**)

Observe that it might be necessary to change the initial mapping of the vertices of 4-cycles in order to make the vertices containing $U_1 \cap V_1$ map to the vertices containing $U_2 \cap V_2$. If none of the constructed two permutations α gives an isomorphism between the corresponding two D-families of type (5), then an isomorphism should be looked for among the two permutations satisfing (**) and

$\alpha(X_1 \cap Y_1) = Y_2 \cap Z_2$; $\alpha(Y_1 \cap Z_1) = Z_2 \cap T_2$ (***)

$\alpha(Z_1 \cap T_1) = T_2 \cap X_2$; $\alpha(T_1 \cap X_1) = X_2 \cap Y_2$; $\alpha(X_1 \cap Z_1) = Y_2 \cap T_2$

Type (6): We denote the additional edges of G_1 and G_2

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by U_1V_1, U_1W_1 and U_2V_2, U_2W_2 respectively.

An isomorphism between the corresponding D-families of type (6) should be realized by the permutation α of S satisfying $\alpha(U_1) = U_2$ and either the conditions (*) or the conditions (***).

This permutation maps $U_1 \setminus (W_1 \cup U_1)$ to $U_2 \setminus (W_2 \cup V_2)$. The elements of the last two sets correspond to two pairs of pairs of opposite edges of the 4-cycles.

Types (33) and (34): We denote (in both cases) the edges of D-graphs G_i $(i=1,2)$ by X_iY_i, X_iZ_i, U_iV_i.

An isomorphism between the corresponding D-families can be realized by a permutation α of S satisfying the following conditions:

$$\alpha(X_1 \cap Y_1) = X_2 \cap Y_2 \quad ; \quad \alpha(S \setminus (X_1 \cup Y_1)) = S \setminus (X_2 \cup Y_2)$$

$$\alpha(X_1 \cap Z_1) = X_2 \cap Z_2 \quad ; \quad \alpha(S \setminus (X_1 \cup Z_1)) = S \setminus (X_2 \cup Z_2)$$

$$\alpha(U_1 \cap V_1) = U_2 \cap V_2 \quad ; \quad \alpha(S \setminus (U_1 \cup V_1)) = S \setminus (U_2 \cup V_2)$$

Type (35): We notice that each vertex of degree 2 contains exactly one intersection element, which corresponds to an edge of the other connected component.

We denote the edges of D-graphs G_i $(i=1,2)$ by $X_iY_i, X_iZ_i, U_iV_i, U_iW_i$ so that $U_i \supset X_i \cap Y_i$, $X_i \supset U_i \cap V_i$

The only permutation α of S, which satisfies the six conditions from the previous case, and, in addition,

$$\alpha(U_1 \cap W_1) = U_2 \cap W_2 \quad ; \quad \alpha(S \setminus (U_1 \cup W_1)) = S \setminus (U_2 \cup W_2)$$

induces an isomorphism between the corresponding D-families.

This completes the proof of Theorem 2.

Theorem 2 ensures that all non-isomorphic D-families are covered by our "example construction". It gives that each example F_o represents one class F_o of isomorphic D-families. The permitted 4-sets for F_o represent the permitted 4-sets for an arbitrary family of F_o. The non-isomorphic possibilities for the augmentation of F_o represent the non-isomorphic possibilities

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for the augmentation of F_o, that is, all the non-isomorphic D-families, which contain a subfamily from F_o.

CONSTRUCTION OF THE REMAINING NON-ISOMORPHIC C-FAMILIES

Given a D-family, we construct all the non-isomorphic C-families having it as the (unique) maximal D-subfamily. We shall apply this construction to the already constructed examples ((1)-(36)) of D-families and so obtain all the non-isomorphic C-families, because these examples represent all the non-isomorphic D-families.

We shall primarily find the 2-permitted 4-sets for all 36 examples of D-families. The 2-permitted sets necessarily arise in complementary pairs (because the complement of a 2-permitted set is 2-permitted).

The D-families of types
(12), (15), (17), (18), (19), (24), (26), (27), (28), (30), (31), (32), (36)
have no 2-permitted 4-sets and they coincide with the only corresponding C-families.

We denote the complementary pairs

1258-3467, 1268-3457, 1278-3456		a,b,c
1358-2467, 1368-2457, 1378-2456	by	d,e,f respectively
1458-2367, 1468-2357, 1478-2356		g,h,i

The complementary pairs of 2-permitted 4-sets for the given examples of the remaining types of D-families are given in the following list (the types are denoted in the upper row):

(1)	(2)	(3)	(4)	(5)	(6)	(7)	(8)
a,b,c,d,e,f,g,h,i	c,d,e,g,h	e,f	c,d,e,g,h	c	c	c	c,e,g

(9)	(10)	(11)	(13)	(14)	(16)	(20)	(21)	(22)	(23)	(25)	(29)	(33)	(34)	(35)
c,e,g	c	c	c	c	c	g,h	g,h	g,h	g	g,h	g	c	c	c

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THEOREM 3. *All the non-isomorphic C-families, which are not D-families, can be constructed from the given examples of D-families by the addition of some 2-permitted 4-sets from the pairs c,e and g.*

P r o o f. Due to the above list, it suffices to give proof for the types (1),(2),(3),(4),(20),(21),(22),(25).

We observe that no two 4-sets, which belong to two different complementary pairs, appearing in any of the following 3-sets of complementary pairs

{a,b,c},{d,e,f},{g,h,i},{a,d,g},{b,e,h},{c,f,i}

may be added to a D-family, otherwise either 3-intersections or new 1-intersections appear.

Equivalently, the only maximal sets of complementary pairs of 4-sets, the 4-sets from which may be added simultaneously to a D-family are

{a,e,i},{a,f,h},{b,d,i},{b,f,g},{c,d,h},{c,e,g}

Each of these six may appear with the D-family (1). Note, however, that the following permutations of S, denoted just by their non-trivial cycles

(57), (567), (576), (67), (56)

in this order, map the 4-sets appearing in {c,e,g} to the 4-sets in the corresponding one of the first five 3-sets of pairs. The elements 5,6,7 are in the equal position with respect to the given example of type (1) (all the three appear in the same set, 1567). Thus the addition of the 4-sets, contained in anyone of the first five 3-sets of pairs to 1234,1567 gives the isomorphic C-families with those obtained by addition to 1234,1567 just some of the 4-sets contained in c,e or g.

The transposition (56) maps {e,g} to {d,h} and {g} to {h}. The elements 5 and 6 always appear together in the sets of the given examples of D-families of types (2),(4),(20),(21),(22),(25). Consequently, the 2-permitted 4-sets from the families {d,h}, respectively {h}, can be always (when these six D-families are considered and when only non-isomorphic C-families are looked for) replaced by the 2-permitted 4-sets from the families {e,g}, respectively {g}.

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Similarly, the 2-permitted 4-sets from {f}, with respect to the given example of D-family of type (3), can be always replaced by 2-permitted 4-sets from {e}, due to the transposition (67).Q.E.D.

The given examples of D-families of types (3),(5),(6),(7),(10),(11),(13),(14),(16),(20),(21),(22),(23),(25),(29),(33),(34),(35) have 2-permitted 4-sets in just one of the pairs c,e,g. There are at least three non-isomorphic C-families, corresponding to each of these D-families; the families obtained by addition of none, one and two 2-permitted 4-sets.

The only question which should be answered here is: "In which of these 18 cases does there exist the fourth non-isomorphic C-family?", or, equivalently
"In which of these 18 cases are the two 2-permitted 4-sets not in the equal position with respect to the D-family?"

It is not hard to check that the answer is:
"With D-families of types (5),(6),(10),(11),(13),(14),(20),(23),(33),(34)".

For example, we show that the 4-sets 1458 and 2367 are in the equal position with respect to the given example H_2 of the D-family of type (22), but not with respect to the given example H_3 of the D-family of type (23).

The permutation $\begin{pmatrix}1&2&3&4&5&6&7&8\\3&8&1&7&6&5&4&2\end{pmatrix}$ maps $H_2 \cup \{1458\}$ to $H_2 \cup \{2367\}$.

The elements 1 and 7 have special, but different positions in the family H_3. Both are the intersection elements corresponding to the outer edges of the D-graph (4-path). However, 7 is missing in two 4-sets, which determine an edge, but 1 is not (equivalently, the elements 1 and 7 are in different positions with respect to the markation). Consequently, the 4-sets 1458 (containing 1) and 2367 (containing 7) are not in the equal position with respect to the family H_3.

THEOREM 4. *There is a bijection between the non-isomorphic C-families, arising from the given examples F_0 and F_1 of the D-families of types (2) and (4) respectively.*

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P r o o f. As $3478 = (1234 \setminus (1567 \cup 2568)) +$
$+ (((1567 \cup 2568) \setminus (1567 \cap 2568)) \setminus 1234)$
and any isomorphism between two C-families arising from F_o preserves both 1234 and $\{1567, 2568\}$, we conclude that it also preserves 3478, that is, it is also an isomorphism between the two C-families arising from F_1 and having the same 2-permitted 4-sets as the first two C-families (Note that the existence of the later two C-families is guaranteed).

Conversely, any isomorphism between two C-families arising from F_1 induces (as a restriction) an isomorphism between their C-subfamilies, obtained by deleting the same 4-set of F_1, from each of the first two C-families. These C-subfamilies have the maximal D-subfamilies of type (2) and hence each of them is isomorphic to a C-family arising from F_o.

The only non-isomorphic C-families, which are not covered by the discussion above, are those C-families, the maximal D-subfamilies of which have stars as the corresponding D-graphs (types (1),(2),(8),(9)).

The 4-set 1234 is the centre of the given examples of stars corresponding to D-families of types (2),(8),(9) and we adopt the convention that it is also the centre in the case of type (1). Each of the 4-sets from the pairs 1278-3456,1368-2457, 1458-2367 (that is, from the pairs of c,e,g) has a 2-intersection with 1234 and no two of them have the same 2-intersection with 1234.

THEOREM 5. *There is a bijection between the non-isomorphic C-families arising from the given examples of D-families of types (1),(2),(8),(9) and the non-isomorphic coloured C-graphs*(*) *having one, two, three, four black vertices respectively.*

P r o o f. We adjoin C-graphs to these C-families as follows:

The four vertices of the C-graphs are denoted by 1,2,3,4. The vertex 1 (resp.2,resp.3,resp.4) is black if and only if the corresponding C-family contains the set 1567 (resp.2568,resp. 3578,resp.4678). The edge 12 (resp.13,resp.14,resp.23,resp.24, resp.34) exists if and only if the corresponding C-family

(*) Two coloured C-graphs G_1 and G_2 are isomorphic if there is a graph-isomorphism which maps G_1 onto G_2 and preserves the colouring.

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contains the 4-set 1278 (resp.1368,resp.1458,resp.2367,resp. 2457,resp.3456)

Conversely, given a coloured C-graph, the vertices of which are denoted by 1,2,3,4 so that the black vertices are denoted by smaller numbers, we simply read all the 4-sets of the corresponding C-family (except for 1234, the existence of which is assumed in advance), just by the use of the above mentioned correspondence.

(I) Suppose that two C-graphs G_1 and G_2 are isomorphic, that is, there exists a colour and incidence-preserving bijection between their vertices (which are denoted by 1,2,3,4).

We extend the corresponding permutation β of $\{1,2,3,4\}$ to a permutation α of S by postulating:

$$\alpha(i) = \beta(i) \quad ; \quad \alpha(9-i) = 9-\beta(i) \quad , \quad \text{for } i=1,2,3,4$$

We claim that the permutation α estabilishes an isomorphism between the C-families F_1 and F_2, corresponding to the graphs G_1 and G_2 respectively.

Both of the families F_1 and F_2 have 1234 and the same sets from $\{1567,2568,3578,4678\}$, by the above given construction of the C-families, corresponding to C-graphs. The permutation α preserves 1234 and also preserves the later 4-sets (not necessarily identically), since it preserves the black vertices.

There is an edge $\{k,f\}$ in G_1 if and only if there exists the edge $\{\alpha(k),\alpha(f)\}$ in G_2. The edges $\{k,f\}$, respectively $\{\alpha(k),\alpha(f)\}$, imply the existence of the 4-sets $\{k,f,9-f,9-k\}$ respectively $\{\alpha(k),\alpha(f),9-\alpha(f),9-\alpha(k)\}$ in the families F_1, respectively F_2.

(II) Conversely, suppose that two C-families K_1 and K_2, which arise from the same one of the given examples of D-families of types (1),(2),(8),(9), are isomorphic, that is, there is a permutation γ of S which maps the set of K_1 onto the sets of K_2. The set 1234 is necessarily preserved under γ in the case of types (2),(8),(9), because of its special position in the D-graph.

We claim that, in the case of types (2),(8),(9), the restriction of the permutation γ to the set $\{1,2,3,4\}$ estabishes an isomorphism between the corresponding C-graphs (with

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vertices denoted by 1,2,3,4)

Proof of the claim:

vertex i is black in $G_1 \Leftrightarrow K_1$ contains $\{i\} \cup (\{5,6,7,8\} \setminus \{9-i\}) \Leftrightarrow K_2$ contains $\{\gamma(i)\} \cup (\{5,6,7,8\} \setminus \{9-\gamma(i)\}) \Leftrightarrow$ vertex $\gamma(i)$ is black in G_2

vertices i,j are incident in $G_1 \Leftrightarrow$ $\Leftrightarrow K_1$ contains $\{i,j\} \cup (\{5,6,7,8\} - \{9-i, 9-j\}) \Leftrightarrow$ $\Leftrightarrow K_2$ contains $\{\gamma(i), \gamma(j)\} \cup (\{5,6,7,8\} - \{9-\gamma(i), 9-\gamma(j)\}) \Leftrightarrow$ $\Leftrightarrow$ vertices $\gamma(i), \gamma(j)$ are incident in G_2

REMARK: We used the fact that all the sets of {1567, 2568,3578,4678}, respectively of {1278,3456,1368,2457,1458, 2367}, can be represented in the first, respectively in the second, of the two forms above.

When the D-family {1234,1567} (of type (1)) is considered, it may happen that the given isomorphism γ between two corresponding C-families K_1, K_2 maps 1234 to 1567 and conversely. It necessarily has 1 and 8 as fixed points.

In this case we define another permutation δ of S by

$\delta(1) = 1$; $\delta(8) = 8$; $\delta(i) = 9-\gamma(i)$ for $2 \leq i \leq 7$

It is easy to check that

$\delta(X) = \gamma(X)$ for each $X \in \{1278,3456,1368,2457,1458,2367\}$

and $\delta(1234) = 1234$ $\delta(1567) = 1567$

It follows that there exists an isomorphism between K_1 and K_2, which preserves 1234 and we may proceed as in the previous three cases.

This completes the proof of Theorem 5.

CONSEQUENCE 1. *There is a bijection between non-isomorphic C-families arising from (the given examples of) D-families of types (1) and (8).*

P r o o f. Follows immediately by interchanging the black and white colour in the vertices of the corresponding coloured C-graphs.

CONSEQUENCE 2. *There are 11 non-isomorphic C-families arising from (the given example of) D-family of type (9).*

P r o o f. All the vertices in the corresponding

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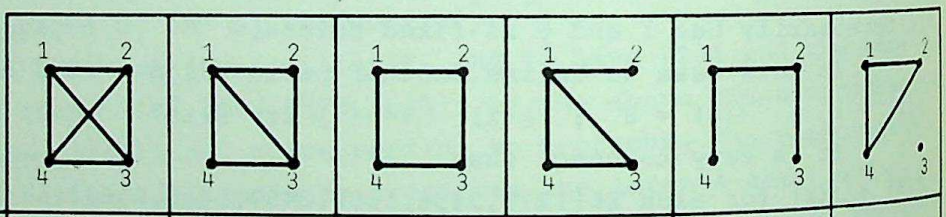
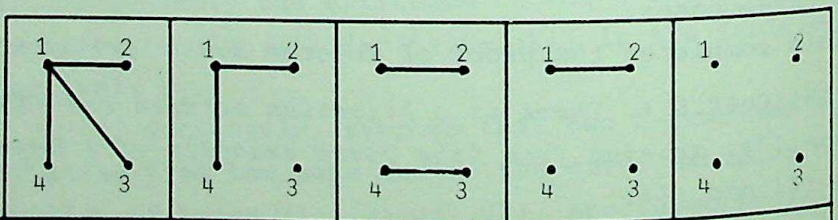
coloured C-graphs are black. We simply use the well-known 11 non-isomorphic simple graphs on 4 vertices (e.g., in the Appendix of |2|)

In order to finish the construction of non-isomorphic C-families, we have just to find all the non-isomorphic 2-colouring of vertices of the above-mentioned 11 graphs, for the cases when exactly one and exactly two vertices are black.

We give the table of non-isomorphic coloured C-graphs with one or two black vertices, by stating their 1-sets and 2-sets of black vertices under the corresponding non-isomorphic graphs on four vertices. If several non-isomorphic colourings correspond to the same graph, then their denotations are separated by commas.

THE TABLE OF NON-ISOMORPHIC COLOURED C-GRAPHS
WITH 1 OR 2 BLACK VERTICES

						
One black vertex is	1	1,2	1	1,2,3	1,3	1,3
two black vertices are	12	12,13,14	12,13	12,13,23,34	12,13,23,34	12,13

					
one black vertex is	1,2	1,2,3	1	1,3	1
two black vertices are	12,23	12,13,23,24	12,13	12,13,34	12

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We stress (somewhat similar as with the denoted A-graphs([1])) that the coloured C-graphs provide immediate constructions of the corresponding, up to isomorphisms determined, C-families with the maximal D-subfamilies of types (1),(2),(4),(8) and (9)

Summing up the numbers of non-isomorphic C-families, arising from all distrinct types of D-families, we obtain that there are 184 non-isomorphic C-families.

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REZIME

NOVA KONSTRUKCIJA PEJVING MATROIDA RANGA 4
NA SKUPU OD 8 elemenata (II)

U ovom radu, koji je nastavak rada {1}, dajemo konstrukciju svih neizomorfnih C-familija (familija 4-podskupova 8-skupa, kod kojih ne postoje 3-preseci, a postoje 1-preseci). Time se kompletira konstrukcija svih neizomorfnih pejving matroida ranga 4 na skupu od 8 elemenata.

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A SEQUENTIAL WEIGHING PROCEDURE

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ABSTRACT

The paper is concerned with a sequential search on a finite set $X = F_1 \cup F_2 \cup \ldots \cup F_n$ of coins, such that $|F_i| = 5$; $i=1,2,\ldots,n$; and $F_i \cap F_j = \emptyset$, for $i \neq j$. There is exactly one counterfeit (heavier) coin in each F_i and we want to identify all of them. A weighing procedure is constructed for an arbitrary n, which implies an upper bound close enough to the theoretical lower bound.

Let $X = F_1 \cup F_2 \cup \ldots \cup F_n$ ($F_i \cap F_j = \emptyset$ if $i \neq j$) be a set of 5n coins, where $F_i = \{c_1^i, c_2^i, c_3^i, c_4^i, c_5^i\}$ is a set containing 5 coins ($i=1,2,\ldots,n$). All coins are indistinguishable except that exactly n of them are slightly heavier than the rest and each of the sets F_i contains exactly one heavier coin. Given a balance scale, we want to find an optimal weighing procedure i.e. a procedure which minimizes the maximum number of steps (weighings) which are required to identify all the heavier coins.

We suppose that all the heavier coins are of equal weight, and so are all the light coins. If w is the weight of a light coin, then the weight of a heavy coin is less than $\frac{n+1}{n} w$, so that the larger of the two numerically unequal subsets of X is always the heavier. This means that no information is gained by balancing two numerically unequal sets. We also suppose

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that the scale reveals which, if either, of the subsets of X is heavier but not by how much.

Some other problems of this type are discussed in greater detail in |1|, |2|, |3|, |4|, |5| and |6|.

Consider a pair (A,B) of numerically equal disjoint subsets of X. Step (A,B) will mean the balancing of A against B. If $A=\{a\}$ and $B=\{b\}$, we write simply (a,b). The possible outcomes are:

(a) The sets balance, symbolized by $A=B$;

(b) The sets do not balance, in which case we use the notation $A>B$, $B>A$, where $>$ between two sets means "is heavier than".

We denote with $\ell_5(n)$ the maximum number of weighings in an optimal weighing procedure for the set X. By information-theoretical arguments, we have the following lower bound for $\ell_5(n)$:

$$\ell_5(n) \geq \lceil n \log_3 5 \rceil \tag{1}$$

where $\lceil x \rceil$ denotes the smallest integer $\geq x$.

Now, we are going to construct a weighing procedure S_n for an arbitrary n, which implies an upper bound close enough to the theoretical lower bound, i.e. we are going to prove the following statement:

THEOREM. $$\ell_5(n) \leq \left\lceil \frac{3n}{2} \right\rceil \tag{2}$$

P r o o f. We shall construct a weighing procedure S_n for the set X, with the maximum number of weighings $\left\lceil \frac{3n}{2} \right\rceil$. This procedure will be sequential i.e. the next pair of subsets to be compared is dependent on the answers to previous weighings.

For $n=1$, the procedure S_1 consists of two independent weighings:

1. (c_1^1, c_2^1) 2. (c_3^1, c_4^1)

It can be easily checked that the heavy coin is determined by the answers to the weighings 1. and 2. according to

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the given TABLE.

TABLE

Answers to the weighings		Heavy coin
1.	2.	
<	<	outcomes impossible
<	=	c_2^1
<	>	outcomes impossible
=	<	c_4^1
=	=	c_5^1
=	>	c_1^1
>	<	outcomes impossible
>	=	c_3^1
>	>	outcomes impossible

For $n = 2$, we construct a sequential weighing procedure S_2 as follows:

STEP 1. (A,B), where $A = \{a_1^1, a_2^1, a_3^1\}$, $B = \{a_4^1, a_1^2, a_2^2\}$.

(i) If $A > B$, this means that each of the sets A and $B' = \{a_3^2, a_4^2, a_5^2\}$ contains exactly one heavy coin and to identify them we can use two independent weighings, (a_1^1, a_2^1) and (a_3^2, a_4^2), respectively.

(ii) If $A = B$, we go to Step 2.

STEP 2. (a_1^2, a_2^2), and now

(1) If $a_1^2 > a_2^2$, the heavy coins are a_1^2 and one coin from the set A, which can be determined by one weighing e.g. (a_1^1, a_2^1);

(2) If $a_1^2 = a_2^2$, the heavy coins are a_5^1 and one coin from the set B', which can be determined by one weighing e.g. (a_3^2, a_4^2);

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(3) If $a_1^2 < a_2^2$, the heavy coins are a_2^2 and one coin from the set A, and we continue as in case (1).

(iii) If $A < B$, we also go to Step 2, and

(1) If $a_1^2 > a_2^2$, the heavy coins are a_1^2 and one of the coins a_4^1 and a_5^1, which can be determined by the balancing of a_4^1 against a_5^1 ;

(2) If $a_1^2 = a_2^2$, the heavy coins are a_4^1 and one coin from the set B´, and we continue as in case (2) for $A = B$;

(3) If $a_1^2 < a_2^2$, the heavy coins are a_2^2 and one of the coins a_4^1 and a_5^1, and we continue as in case (1).

It follows from (1), that the constructed procedures S_1 and S_2 are optimal.

Let $n > 2$. Now, if n is even, we partition the set X into the subsets $X_i = F_{2i-1} \cup F_{2i}$; $i=1,2,\ldots,\lfloor \frac{n}{2} \rfloor$. If n is odd, we partition the set X into the sets $X_i = F_{2i-1} \cup F_{2i}$; $i=1,2,\ldots,\lfloor \frac{n}{2} \rfloor$, and the set $X_{\lfloor \frac{n}{2} \rfloor + 1} = F_n$. Here, $\lfloor x \rfloor$ denotes the greatest integer $\leq x$. We sort each subset X_i $(i=1,2,\ldots,\lfloor \frac{n}{2} \rfloor)$ by applying the optimal procedure S_2 and $X_{\lfloor \frac{n}{2} \rfloor + 1}$ by applying the procedure S_1. So, a procedure S_n, for which the maximum number of weighings is $\lceil \frac{3n}{2} \rceil$, is constructed. Hence follows (2).

The constructed procedure S_n is optimal at least for those n´s for which the upper bound $\lceil \frac{3n}{2} \rceil$ coincides with the information - theoretical lower bound. So, as a consequence of the Theorem, we have:

COROLLARY. *The procedure* S_n *is optimal at least for* $n = 1,2,3,4,5,6,7,8,9,10,11,12,13,14,16,18,20,22,24,26,28.$

We think that the procedure S_n is optimal for many other n´s too, but we do not known how to prove it.

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REZIME

JEDNA SEKVENCIJALNA PROCEDURA MERENJA

U radu je konstruisana jedna sekvencijalna procedura za identifikaciju neispravnih (težih) elemenata u skupu $X = F_1 \cup F_2 \cup \ldots \cup F_n$; $|F_i| = 5$, $i=1,2,\ldots,n$, i $F_i \cap F_j = \emptyset$ za $i \neq j$, gde je n proizvoljan ceo pozitivan broj. Na taj način dobijena je jedna gornja granica za maksimalan broj koraka optimalne procedure, koja je vrlo bliska teorijskoj donjoj granici.

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COUNTING BINARY GRIDS

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ABSTRACT

A binary (m,n)-grid is defined as an array of m rows and n columns formed from mn square cells each of which is crossed by a diagonal. Two (m,n)-grids are said to be equivalent iff they can be transformed one into the other by rigid motion in the space. In this paper the number N(m,n) of non-equivalent (m,n)-grids are determined for arbitrary natural numbers m and n. The formula of N.Hoffman for N(1,n) appears as a special case of our result.

1. INTRODUCTION

A binary (m,n)-grid is defined as an array of m rows and n columns formed from mn square cells each of which is crossed by a diagonal. Thus, Fig. 1 is a (3,4)-grid.

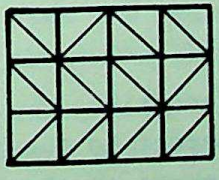

Fig. 1.

Two (m,n)-grids are said to be equivalent iff they can be transformed one into the other by a rigid motion in space.

J.A. Dunn posed the problem of determining the number N(m,n) of non-equivalent (m,n)-grids. Clearly, N(m,n) = N(n,m). N.Hoffman in [1] proved that

$$N(1,n) = 2^{n-2} + 2^{\left\lfloor \frac{n}{2} \right\rfloor - 1} . \tag{1}$$

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In this paper we shall determine the number $N(m,n)$, for arbitrary natural numbers m and n; the Hoffman´s formula appears as a special case of our result.

Let us introduce some notations. By $Y_{(m,n)}$ we shall denote the set of all (m,n)-grids. Once m and n are specified, we shall write simply Y instead of $Y_{(m,n)}$. If A is a set, then the cardinality of A is $|A|$. By $\lfloor x \rfloor$ and $\lceil x \rceil$ we shall denote the smallest integer $\geq x$ and the greatest integer $\leq x$, respectively. It is clear that

$$|Y_{(m,n)}| = 2^{mn} \quad . \tag{2}$$

2. RECTANGULAR (m,n)-GRIDS ($m \neq n$)

Any rigid motion in space by which an (m,n)-grid can be transformed into some other (may be the same) (m,n)-grid - reduces to one of the transformations from the group of symmetries of the rectangle (Fig. 2):

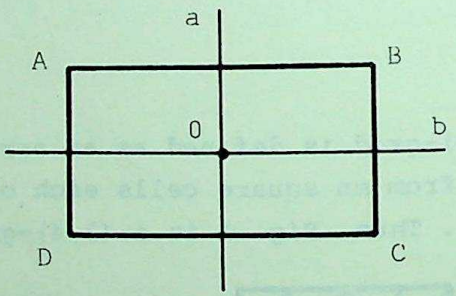

Fig. 2.

i - identical transformation,
a - symmetry with respect to the vertical axis,
b - symmetry with respect to the horizontal axis,
c - symmetry with respect to the center 0 of the rectangle (central symmetry).

Let t(X) denote the grid into which the grid X is transformed by applying the transformation t. We shall consider the following subset of X.

$$A = \{X/a(X) = X\},$$
$$B = \{X/b(X) = X\},$$
$$C = \{X/c(X) = X\}.$$

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Using Burnside's lemma, we obtain:

LEMMA 2.1. *If* $m \neq n$, *then*

$$N(m,n) = \frac{1}{4}\,(|Y| + |A| + |B| + |C|). \tag{3}$$

3. SQUARE GRIDS ((n,n)-GRIDS)

In addition to i, a, b and c, the group of rigid motions which transforms a square into itself, contains four additional transformations (Fig. 3):

d - rotation about the center 0 of square through angle $\phi = 90^{\circ}$,

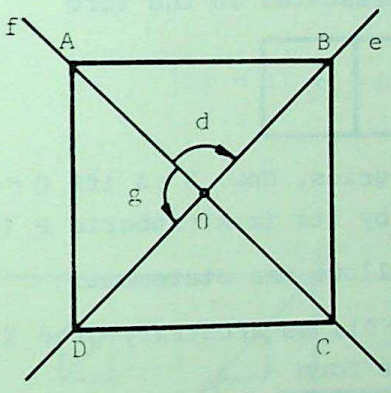

Fig. 3.

e - symmetry with the respect to the diagonal BD,

f - symmetry with the respect to the diagonal AC,

g - rotation about the center 0 through angle $\phi = -90^{\circ}$.

Let A,B,C be subsets of $Y = Y_{(n,n)}$, defined as in Section 2. We shall consider also the following subsets of X:

$$D = \{X/d(X) = X\};$$
$$F = \{X/f(X) = X\};$$
$$G = \{X/g(X) = X\}.$$

Using Burnside's lemma, we obtain:

LEMMA 3.1.

$$N(n,n) = \frac{1}{4}\,|D| + \frac{1}{8}\,(|Y|+|A|+|B|+|C|+|E|+|F|). \tag{4}$$

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4. THE THEOREM

Before stating the theorem, we shall prove some lemmas.

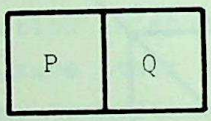
LEMMA 4.1. *For arbitrary natural* m,n, *and for the corresponding subsets* A,B,C *of* $Y_{(m,n)} = Y$

(5) (i) $$|A| = (1 + (-1)^n)\, 2^{\frac{mn}{2} - 1},$$

(6) (ii) $$|B| = (1 + (-1)^m)\, 2^{\frac{mn}{2} - 1},$$

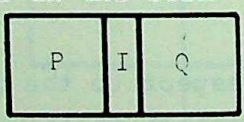
(7) (iii) $$|C| = 2^{\left\lceil \frac{mn}{2} \right\rceil}.$$

P r o o f. (i) For $n = 2k$ $(k \in N)$, an arbitrary grid X from $Y_{(m,n)}$ can be represented in the form



where P and Q are (m,k)-grids. Now, $X \in A$ iff $Q = a(P)$. It means that $X \in A$ is determined by its (m,k)-subgrid P (containing $mk = \frac{mn}{2}$ cells), hence follows the statement.

For $n = 2k + 1$ $(k \geq 0)$, an arbitrary grid X from $Y_{(m,n)}$ can be represented in the form



where P and Q are (m,k)-grids and I is a (m,1)-grid. Now, $X \in A$ iff $Q = a(P)$ and $I = a(I)$. However, $a(I) \neq I$, hence $A = \emptyset$, i.e. $|A| = 0$, and (5) is proved.

(ii) (6) can be proved in the same way.

(iii) $X \in C$ iff any two cells situated symmetrically with respect to the center 0 of X are of the same sort (either ▨ or ▧). If mn is even, $X \in C$ is determined by $\frac{mn}{2} = \left\lceil \frac{mn}{2} \right\rceil$ pairs of cells. If both m and n are odd, the central cell is symmetrical to itself and it can be of an arbitrary sort. In that case, $X \in C$ is determined by $\frac{mn-1}{2} + 1 = \left\lceil \frac{mn}{2} \right\rceil$ pairs of cells (central cell included). Hence, follows the statement.

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REMARK.

If $m = n$, (5), (6) and (7) become:

$$|A| = |B| = (1 + (-1)^n)\, 2^{\frac{n^2}{2} - 1} \tag{8}$$

$$|C| = 2^{\frac{n^2}{2}} \tag{9}$$

LEMMA 4.2. *For arbitrary* $n \in N$, *and for corresponding subsets* D, E, F of $Y_{(m,n)} = Y$:

$$|E| = |F| = 2^{\frac{n^2+n}{2}}, \tag{10}$$ (i)

$$|D| = (1 + (-1)^n)\, 2^{\frac{n^2}{4} - 1}. \tag{11}$$ (ii)

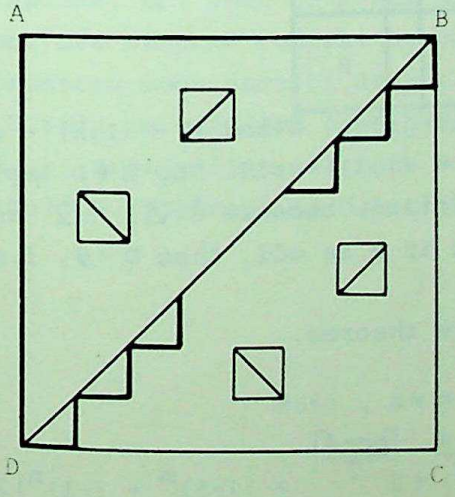

Fig. 4.

P r o o f. (i) $X \in E$ iff any two cells, situated symmetrycally with respect to the diagonal BD (Fig. 4), are of the same sort (either ⧄ or ⧅). Each cell situated on the diagonal BD may be of an arbitrary sort (either ⧄ or ⧅). This means that $X \in E$ is determined by its cells situated on the

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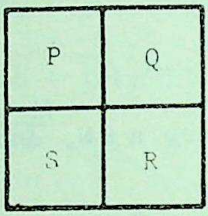
diagonal BD and above it. The number of such cells is $1+2+\dots+n = \frac{n^2+n}{2}$. Hence it follows that $|E| = 2^{(n^2+n)/2}$.

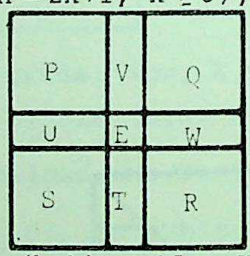
In the same way we can prove that $|F| = 2^{(n^2+n)/2}$.

(ii) If n is even ($n = 2k$, $k \geq 1$), then $X \in Y_{(n,n)}$ can be represented in the form



where P, Q, R and S are (k,k)-grids. Now, $X \in D$ iff $Q = d(P)$, $R = d(Q) = d^2(P)$ and $S = d(R) = d^3(P)$. This means that $X \in D$ is determined by its subgrids P containing $n^2/4$ cells, hence follows the statement.

If n is odd ($n = 2k+1$, $k \geq 0$), then $X \in Y_{(n,n)}$ can be represented in the form



where P, Q, R and S are (k,k)-grids, U and W - (1,k) - grids, V and T - (k,1)-grids and E - (1,1) -grid. Now $X \in D$ implies $d(E) = E$, which is a contradiction, because $d(\boxtimes) = \boxtimes$ and $d(\boxtimes) = \boxtimes$. This means that if n is odd, then $D = \emptyset$, i.e. $|D| = 0$, and (11) is proved.

Now, we can prove the theorem.

THEOREM. *(i) If* $m \neq n$, *then*

$$N(m,n) = 2^{mn+2} + 2^{\frac{mn-4}{2}} + 2^{\left\lceil \frac{mn-4}{2} \right\rceil} + ((-1)^m + (-1)^n)\, 2^{\frac{mn-6}{2}}; \tag{12}$$

(ii)

$$N(n,n) = 2^{n^2-3} + 2^{\left\lceil \frac{n^2-6}{2} \right\rceil} + 2^{\frac{n^2+n-4}{2}} + (1 + (-1)^n)\,(2^{\frac{n^2-12}{4}} + 2^{\frac{n^2-6}{2}}). \tag{13}$$

P r o o f. (i) Follows from (3), (2), (5), (6) and (7).

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(ii) Follows from (4),(2),(8),(9),(10) and (11).

REMARK

For m = 1, (12) reduces to Hoffman´s formula (1).

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REZIME

PREBROJAVANJE BINARNIH REŠETKI

Binarna (m,n)-rešetka definiše se kao pravougaonik podeljen na mn podudarnih kvadrata, koji su rasporedjeni u m vrsta i n kolona, pri čemu je svaki od njih presečen jednom dijagonalom. Dve binarne rešetke su ekvivalentne ako se kretanjem u prostoru mogu dovesti do poklapanja.

U radu je odredjen broj N(m,n) neekvivalentnih (m,n) -rešetki, za proizvoljne prirodne brojeve m i n, čime je uopštena formula N. Hoffmana za N(1,n).

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A COMBINATORIAL IDENTITY AND ITS APPLICATIONS

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ABSTRACT

An identity which has some interesting combinatorial interpretations is proved. A bijection is established between a set of strings over the alphabet B = {0,1,2,3} and the set of all symmetric monotone functions of n variables over the three-valued logic algebra. As a consequence, a simple formula for the number of such functions is obtained. A different proof of this formula is given in |2|.

1. DEFINITIONS AND NOTATION

Let X denote a finite and nonempty set of symbols; X is called an alphabet. By X^n we shall denote the set of all strings of the length n over the alphabet X, i.e. $X^n = \{x_1x_2 \dots x_n | x_1, x_2, \dots, x_n \in X\}$, the only element of X^0 being the empty string λ, i.e. the string of length 0. The set of all finite strings over the alphabet X is $X^* = \bigcup_{i \geq 0} X^i$.

We shall also use some special notations:

$A = \{0,1\}$;
$B = \{0,1,2,3\}$;
$C = \{1,2\}$;

$\ell_j(a)$ - the number of j's in the string $a \in A^*$, for $j \in A$;
$\ell_j(b)$ - the number of j's in the string $b \in B^*$, for $j \in B$;
$\ell_j(c)$ - the number of j's in the string $c \in C^*$, for $j \in C$;
$K_A(n,s) = \{a | a \in A^{2n}, \ell_1(a) - \ell_0(a) = 2s\} =$
$= \{a | a \in A^{2n}, \ell_1(a) = n+s, \ell_0(a) = n-s\}$;

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$K_B(n,s) = \bigcup_{s \le i \le n} K_B^i(n,s)$, where, for each i such that $s \le i \le n$ and $i - s \equiv 0 \pmod 2$

$$K_B^i(n,s) = \{b \mid b \in B^n,\ \ell_3(b)+\ell_o(b) = i,\ \ell_3(b)-\ell_o(b) = s\} =$$

$$= \{b \mid b \in B^n,\ \ell_3(b) = \frac{i+s}{2},\ \ell_o(b) = \frac{i-s}{2}\}\ ;$$

$$H_B(n) = \bigcup_{i=o}^{n} H_B^i(n), \text{ where, for each } i \text{ such that } 0 \le i \le n$$

$$H_B^i(n) = \{b_1b_2\ldots b_n \mid b_1b_2\ldots b_n \in B^n,\ \ell_1(b_1b_2\ldots b_n) +$$

$$+ \ell_2(b_1b_2\ldots b_n) = i,\ \ell_2(b_1b_2\ldots b_k) \ge$$

$$\ge \ell_1(b_1b_2\ldots b_k) \text{ for each } k \le n\}\ ;$$

$$H_C(i) = \bigcup_{j=o}^{\lfloor \frac{i}{2} \rfloor} H_C^j(i), \text{ where, for each } j \text{ such that } 0 \le j \le \left\lfloor \frac{i}{2} \right\rfloor$$

$$H_C^j(i) = \{c_1c_2\ldots c_i \mid c_1c_2\ldots c_i \in C^i,\ \ell_1(c_1c_2\ldots c_i) = j,$$

$$\ell_2(c_1c_2\ldots c_k) \ge \ell_1(c_1c_2\ldots c_k) \text{ for each } k \le i\}\ .$$

It is obvious, that for $i_1 \neq i_2$:

$$K_B^{i_1}(n,s) \cap K_B^{i_2}(n,s) = \emptyset\ ,$$

$$H_B^{i_1}(n) \cap H_B^{i_2}(n) = \emptyset, \text{ and for } j_1 \neq j_2 :$$

$$H_C^{j_1}(i) \cap H_C^{j_2}(i) = \emptyset\ .$$

If S is a set, then $|S|$ is the cardinality of S. By $\lceil x \rceil$ and $\lfloor x \rfloor$ we denote the smallest integer $\ge x$ and the greatest integer $\le x$, respectively.

2. RESULTS AND DISCUSSION

THEOREM 1. *If* $0 \le s \le n$, *then*

$$(1) \qquad \sum_{i=s}^{n} \binom{n}{i} \binom{i}{\frac{i-s}{2}} 2^{n-i} = \binom{2n+1}{n-s}\ .$$

First, we shall prove the following lemma:

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LEMMA 1. *If* $0 \le s \le n$, *then*

$$(2)\qquad \sum_{\substack{s\le i\le n\\ i-s\equiv 0 \pmod 2}} \binom{n}{i}\binom{i}{\frac{i-s}{2}} 2^{n-i} = \binom{2n}{n-s}.$$

P r o o f. It is obvious that the sets A^{2n} and B^n are of the same cardinality $2^{2n} = 4^n$. An obvious bijection between the sets A^{2n} and B^n is the function $f : A^{2n} \to B^n$ such that for $a = a_1a_2\dots a_{2n} \in A^{2n}$ and $b = b_1b_2\dots b_n \in B^n$, $f(a) = b$ iff for each $k = 1,2,\dots,n$, $\phi(a_{2k-1}a_{2k}) = b_k$, where ϕ is the bijection $\phi = \begin{pmatrix} oo & ol & lo & ll \\ o & 1 & 2 & 3 \end{pmatrix}$ between the sets A^2 and B. It is easy to see that $a \in K_A(n,s) \subset A^{2n}$ iff $b \in K_B(n,s) \subset B^n$, where $b = f(a)$. It means that the restriction of f to $K_A(n,s) \subset A^{2n}$ is a bijection between the sets $K_A(n,s)$ and $K_B(n,s)$, i.e. $|K_A(n,s)| = |K_B(n,s)|$. On the other hand

$$(3)\qquad |K_A(n,s)| = \binom{2n}{n-s} \quad \text{and}$$

$$(4)\qquad |K_B(n,s)| = \sum_{\substack{s\le i\le n\\ i-s\equiv o \pmod 2}} \binom{n}{i}\binom{i}{\frac{i-s}{2}} 2^{n-i}.$$

The last equality follows from the fact that for each i such that $s \le i \le n$ and $i-s \equiv 0 \pmod 2$:

$$|K_B^i(n,s)| = \binom{n}{i}\binom{i}{\frac{i-s}{2}} 2^{n-i},$$

which can be easily proved.

Now, (2) follows from (3) and (4).

P r o o f of Theorem 1. Substituting $s+1$ instead of s in (2), we obtain

$$\sum_{\substack{s+1\le i\le n\\ i-s-1\equiv 0 \pmod 2}} \binom{n}{i}\binom{i}{\frac{i-s-1}{2}} 2^{n-i} = \binom{2n}{n-s-1}$$

i.e.

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$$(5)\qquad \sum_{\substack{s+1\le i\le n \\ i-s\equiv 1 \pmod 2}} \binom{n}{i} \binom{i}{\frac{i-s-1}{2}} 2^{n-1} = \binom{2n}{n-s-1} .$$

Summing up(2) and (5), we obtain (1).

A combinatorial interpretation of (1). The number of strings of the length 2n+1 over the alphabet {x,y}, for which the difference between the number of x´s and the number of y´s is 2s+1, equals the number of strings of the length n over the alphabet {x,y,z,u} for which the difference between the number of x´s and the number of y´s is either s or s+1.

For s=0, we have

COROLLARY.

$$(6)\qquad \sum_{i=0}^{n} \binom{n}{i} \binom{i}{\lfloor \frac{i}{2} \rfloor} 2^{n-i} = \binom{2n+1}{n} .$$

A combinatoral interpretation of (6). The number of strings of the length 2n+1, over the alphabet {x,y} with n+1 x´s and n y´s, equals the number of strings of the length n over the alphabet {x,y,z,u} with $\lceil \frac{i}{2} \rceil$ x´s and $\lfloor \frac{i}{2} \rfloor$ y´s, for some i=0,1,...,n.

3. APPLICATIONS

First, we use (6) to count the number of strings in the set $H_B(n)$.

THEOREM 2. $|H_B(n)| = \binom{2n+1}{n}$.

First, we shall prove the following lemma:

LEMMA 2. $|H_C(i)| = \binom{i}{\lfloor \frac{i}{2} \rfloor}$.

P r o o f. It is known (see |1|, pp. 65-66) that for $j \le \lfloor \frac{i}{2} \rfloor$:

$$|H_C^j(i)| = \binom{i}{i-j} - \binom{i}{i-j+1} .$$

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Hence, it follows that

$$|H_C(i)| = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} |H_C^j(i)| = \sum_{j=0}^{\lfloor \frac{i}{2} \rfloor} \left(\binom{i}{i-j} - \binom{i}{i-j+1} \right) = \binom{i}{\lfloor \frac{i}{2} \rfloor} .$$

P r o o f of Theorem 2. It is easy to see that

$$|H_B^i(n)| = \binom{n}{i} |H_C(i)| 2^{n-1} = \binom{n}{i} \binom{i}{\lfloor \frac{i}{2} \rfloor} 2^{n-i} .$$

Now, we have, by using Lemma 2 and Corollary:

$$|H_B(n)| = \sum_{i=0}^{n} |H_B^i(n)| = \sum_{i=0}^{n} \binom{n}{i} \binom{i}{\lfloor \frac{i}{2} \rfloor} 2^{n-i} = \binom{2n+1}{n} .$$

Now, we are going to determine the number of symmetric monotone functions of n variables over the three - valued logic algebra i.e. the number of functions $F : E_3^n \to E_3$, where $E_3 = \{0,1,2\}$, and the following two conditions are satisfied (|3|):

(i) $(x_1,x_2,\ldots,x_n) \leq (y_1,y_2,\ldots,y_n)$ implies

$F(x_1,x_2,\ldots,x_n) \leq F(y_1,y_2,\ldots,y_n)$,

under the assumption that $0 < 1 < 2$ and

$(x_1,x_2,\ldots,x_n) \leq (y_1,y_2,\ldots,y_n)$ iff $x_i \leq y_i$ for all $i=1,2,\ldots,n$.

(ii) $F(x_{i_1},x_{i_2},\ldots,x_{i_n}) = F(x_1,x_2,\ldots,x_n)$, for each permutation $(x_{i_1},x_{i_2},\ldots,x_{i_n})$ of $(x_1,x_2,\ldots,x_n)$.

Since F is a symmetric function, the set of all n-tuples with p 0´s, q 2´s and n-p-q 1´s can be represented by the n-tuple $(\underbrace{0,\ldots,0}_{p},1,\ldots,1,\underbrace{2,\ldots,2}_{q})$ which we denote, shortly, by (p,q). So there is a bijection between the set of all symmetric monotone functions $F : E_3^n \to E_3$ and the set of all monotone functions $F' : L_{n+1}^{(2)} \to E_3$, where $L_{n+1}^{(2)}$ is the set of all

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ordered pairs (p,q) of integers such that $p \geq 0$, $q \geq 0$, $p+q \leq n$, and $(p,q) \leq (p',q')$ iff $p \geq p'$ and $q \leq q'$. The set $L^{(2)}_{n+1}$ partially order by the relation $\leq$ can be represented as a lattice on the Cartesian plane.

Figure 1. is such a lattice for $n = 6$.

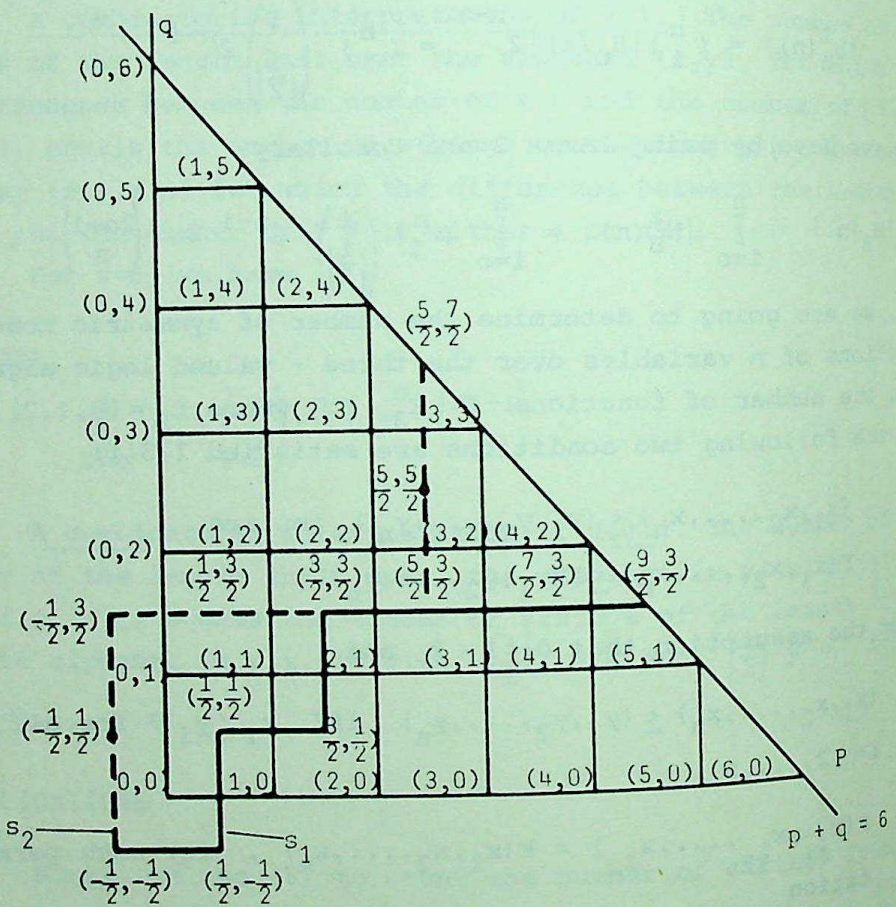

Figure 1.

Any function $F' : L^{(2)}_{n+1} \to E_3$ is completely determined by three sets

$$T_i = \{(p,q) \mid (p,q) \in L^{(2)}_{n+1}, \ F'(p,q) = i\}, \quad \text{for } i=0,1,2.$$

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We also consider the lattice of all points $(p-\frac{1}{2}, q-\frac{1}{2})$, such that $p \geq 0$, $q \geq 0$, $p+q \leq n+1$. An increasing path from $(-\frac{1}{2}, -\frac{1}{2})$ is a set of edges of this lattice which at each point increases in p or increases in q.

All increasing paths of n+1 edges which begin at $(-\frac{1}{2}, -\frac{1}{2})$ must end somewhere on the line $p+q=n$ (indicated in Figure 1 for $n=6$). Label each edge of such a path by 0 if it increases in p and by 1 if it increases in q. So, there is a bijection between the set of all increasing paths of n+1 edges which begin at $(-\frac{1}{2}, -\frac{1}{2})$ and the set A^{n+1}. In Figure 1, two such paths s_1 and s_2, for $n=6$, are drawn and corresponding strings are 0101000 and 1100011, respectively.

THEOREM 3. *There are* $\binom{2n+3}{n+1}$ *symmetric monotone functions of* n *variables over the three-valued logic algebra.*

P r o o f. It is sufficient to determine the number of all monotone functions $F' : L^{(2)}_{n+1} \to E_3$. However, the sets T_0, T_1 and T_2 for such a monotone function are separated by two increasing paths s_1 and s_2 of n+1 edges beginning at $(-\frac{1}{2}, -\frac{1}{2})$ and such that none of the points of s_2 are below s_1. On the other hand, such two paths always determine a monotone function $F' : L^{(2)}_{n+1} \to E_3$ by specifying corresponding sets T_0, T_1 and T_2.

So, there is a bijection between the set of all symmetric monotone functions $F : E_3^n \to E_3$ and the set of all pairs of strings $a_1a_2\ldots a_{n+1}$, $a'_1a'_2\ldots a'_{n+1} \in A^{n+1}$ such that $\ell_1(a_1a_2\ldots a_k) \leq \ell_1(a'_1a'_2\ldots a'_k)$ for each $k=1,2,\ldots,n+1$. But, the number of such pairs of strings equals the number $|H_B(n+1)|$ of strings $b_1b_2\ldots b_{n+1} \in B^{n+1}$ such that $\ell_1(b_1b_2\ldots b_k) \leq \ell_2(b_1b_2\ldots b_k)$ for each $k=1,2,\ldots,n+1$; a corresponding bijection can be constructed by taking $b_i = 2a'_i + a_i$. From Theorem 2 it follows that this number is $\binom{2n+3}{n+1}$.

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REZIME

JEDAN KOMBINATORNI IDENTITET I NJEGOVE PRIMENE

U radu se dokazuje jedan identitet (formula (1)) koji ima interesantne kombinatorne interpretacije. Uspostavljenja je bijekcija izmedju jednog skupa reči nad četvoroelementnom azbukom i skupa svih simetričnih monotonih funkcija od n promenljivih troznačne logike. Kao posledica dobijena je formula za broj tih funkcija - dokazano je da ih ima $\binom{2n+3}{n+1}$.

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A NOTE ON FORCING AND WEAK INTERPOLATION THEOREM FOR INFINITARY LOGICS

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ABSTRACT

Our considerations are connected with the results of the first three chapters of |1|. The aim of this paper is to contrubute in some way to a better understanding of the purpose of introducing weak formulas while dealing with the forcing relation for infinitary logics (Theorem 1.23) as well as to correct, in our opinion, the proof of the weak form of the Interpolation theorem for infinitary logics.

INTRODUCTION

It is already announced in |1| that the proofs of: for each p $p \Vdash \sim\sim PC_{\infty}1$ or of preserving E_{∞} "seems to involve some kind of saturation property for C" i.e. (*) for each p from $p \Vdash \wedge\sim\sim \Phi$ follows $p \Vdash \sim\sim \wedge \Phi$ (we are always given the other implication). We have shown that all these statements are in fact equivalent (thus mutually equivalent) to: for each p (and ϕ) $p \Vdash \sim\sim \phi$ iff $p \Vdash \phi^{wk}$ (Theorem 1.23). A sufficient condition, merely conjectured in |1|, that these statements hold, is that any nondecreasing sequence of length $\alpha < k$, when a fragment of some logic $L_{k\mu}$ is considered, has an upper bound (Lemma 1.11). But this is not a necessary condition too (example 1.13). From (*) follows also: for each p $p \Vdash \sim\sim PC11$ which otherwise when the given logic is infinitary does not have to be fulfilled (Example 1.15).

As for the proof of the Weak Interpolation Theorem for

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infinitary logics (semantical $\models$ is replaced by syntactical $\vdash$) our main objection is that relation ($\|\!\!-$) applied in it does not have to be (and in cases of real interest, is not) a forcing relation, while on the other side the properties of forcing relations, in particular 1.22, are used. All the troubles are overcome successfully by the construction of a forcing relation which has a "nice" intersection with the given one (Lemmas 2.12 - 2.23, Theorem 2.24).

Some other corrections and a few, we hope, useful remarks are made.

§ 0. We shall assume a knowledge of the basic properties of a forcing relation and in particular a familiarity with |1|. However for the reader's convenience we shall cite some of the most relevant definitions and results, mostly from |1|, maybe with some slight, unessential reformulations but using the same terminology and notation.

Through the whole article the language L in question, in all general discussions, will be a first-order language (finitary or not) containing at least one constant symbol. The basic logic symbols will be $\sim$ (negation), & (conjunction), $\exists$ (existential quantifier) and (in the case of infinitary logics) $\bigwedge$ (infinite conjunction). The other like $\vee$ (disjunction), $\rightarrow$ (implication), $\forall$ (universal quantifier) and $\bigvee$ (infinite disjunction) are defined by the basic ones in the standard way. $\bigwedge\Phi$ will replace $\bigwedge_{\phi\in\Phi}\phi$.

We shall just recall that the system of axioms for finitary logic used in |1|, is divided into the groups: (A) substitution instances of propositional tautologies, (B) basic quantificational axioms, (D) generalized quantificational axioms and when the logic is with equality (C) identity axioms. The only rule of inference is modus ponens. For the necessity of infinitary logics we shall redefine in the natural way the axioms of group (D) (and get D_∞), add a new set of axioms (A_∞) $PC_\infty 1 : \bigwedge\sim\sim\phi\rightarrow\bigwedge\phi$, $PC_\infty 2 : \bigwedge\Phi\rightarrow\phi$ for all $\phi\in\Phi$ and one more rule of inference E_∞ : if $\psi\rightarrow\phi$ for all $\phi\in\Phi$ then $\psi\rightarrow\bigwedge\Phi$. Of course, in the formulas of other axiom schemes, formulas

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of infinite length can occur. One can easily see that there a redundancy in the offered system of axioms. So, for instance part (A) of it can be rather restricted and it is obvious that $PC_\infty 2$ and E_∞ with the help of the axioms of part (A) give $PC_\infty 1$. The set of all these axioms and rules of inference will be denoted by Λ_o.

§ 1. Let $<C, \leq, 0>$ be a partially ordered set with the least element 0, AT(L) the set of atomic and SENT(L) the set of all sentences of a language L (we will often write only AT and SENT rather than AT(L) and SENT(L) on the condition that it is clear from the context what is meant by it

DEFINITION 1.1. *A unary relation* $\Vdash$ *on* $C \times SENT(L)$ *is a forcing relation if it satisfies the following conditions (instead of* $(p,\phi) \in \Vdash$ *we shall use the more common* $p \Vdash \phi$ *; of course* $p \nVdash \phi$ *will stand for* $(p,\phi) \notin \Vdash$ *):*

(1) The compability condition(s): :
For each $p,q \in C$ *, for any* $\phi \in AT$
$p \leq q$ *and* $p \Vdash \phi$ *imply* $q \Vdash \phi$ *;*

If L is the language with equality we demand also
(1) (i) For each $p \in C$ *and each closed term t there exists* $q \in C$, $q \geq p$ *and* $q \Vdash t = t$.
(ii) For all closed terms t_1, t_2 *, for any atomic formula* $\phi(v)$ *with at most one variable free and for each* $p \in C$ *there exists* $q \in C$ *such that* $q \geq p$ *and either* $p \nVdash t_1 = t_2$ *or* $p \nVdash \phi(t_1)$ *or* $q \Vdash \phi(t_2)$;
(2) $p \Vdash \phi_1 \& \phi_2$ *if and only if* $p \Vdash \phi_1$ *and* $p \Vdash \phi_2$;

If L is an infinitary language we introduce too
(2) (i) $p \Vdash \Lambda \Phi$ *if and only if* $p \Vdash \phi$ *for each* $\phi \in \Phi$;
(3) $p \Vdash \sim \phi$ *if and only if for each* $q \geq p$ $q \nVdash \phi$
(4) $p \Vdash \exists v \phi(v)$ *if and only if there exists a closed term t such that* $p \Vdash \phi(t)$.

The elements of C will be, as usual, called conditions. We read $p \Vdash \phi$ as p forces ϕ . When $p \Vdash \sim\sim\phi$ we say that (a

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condition) p weakly forces ϕ.

In defining some forcing relation we shall give only its intersection with $C \times AT$ which is obviously sufficient.

DEFINITION 1.2 *A forcing system is a triple* $\langle C, \Vdash, L \rangle$ *where* C *is a partially ordered set with the least element,* L *a given language and* $\Vdash$ *a forcing relation on* $C \times SENT(L)$.

DEFINITION 1.3. *Let* $\langle C, \Vdash, L \rangle$ *be a forcing system where* L *is a finitary logic. For* $p \in C$

$$T^C|p| = \{\phi \in SENT \mid p \Vdash \sim\sim \phi\}$$

Instead of $T^C|0|$, *where* 0 *is the least element of* C *we write just* T^C. T^C *is called the (forcing) companion.*

In any of the propositions that follow, if it is not already written it goes without saying that some forcing system $\langle C, \Vdash, L \rangle$, fixed but without any special characteristics, is given.

The following assertions are direct consequences of definition 1.1.

THEOREM 1.4. *(a) For any conditions* p,q *and for any sentence* ϕ *if* $p \Vdash \phi$ *and* $q \geq p$ *then* $q \Vdash \phi$;

(b) For each $p \in C$ *and for each sentence* ϕ *either* $p \Vdash \phi$ *or* $p \Vdash \sim\phi$;

(c) For each $p \in C$ *and for each sentence* ϕ *there exists a condition* $q \geq p$ *such that* $q \Vdash \phi$ *or* $q \Vdash \sim \phi$.

LEMMA 1.5. *(a)* If $p \Vdash \phi$ then $p \Vdash \sim\sim \phi$

(b) $p \Vdash \sim \phi$ *if and only if* $p \Vdash \sim\sim\phi$

(c) $p \Vdash \sim\sim \& \Phi$ *if and only if* $p \Vdash \& \sim\sim \Phi$

(d) $p \Vdash \sim \exists v \sim \phi$ *if and only if for all closed terms* t $p \Vdash \sim\sim \phi(t)$.

In |1| is given a complete and very exhaustive syntactic proof of

THEOREM 1.6. *Let* $\langle C, \Vdash, L \rangle$ *be a forcing system where* L *is a finitary language (of an arbitrary cardinality). Then for each* $p \in C$

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(1) $T^C|p|$ *is a consistent, deductively closed set* ($T^C|p| \vdash \phi$ *implies* $\phi \in T^C|p|$);

(2) *If* $\phi(v_1,\ldots,v_n)$ *is a logically valid formula (i.e.* $\vdash_L \phi(v_1,\ldots,v_n)$) *then for any closed terms* $t_1,\ldots,t_n$ $\phi(t_1,\ldots,t_n) \in T^C|p|$.

In this place we would like to mention two things. Firstly, (any) condition p really forces, not merely weakly forces, each of the axioms (for a finitary logic). This follows from Lemmas 1.5 and

LEMMA 1.7. $p\|\vdash \sim(\sim\sim\phi \,\&\, \psi)$ *if and only if* $p\|\vdash \sim(\phi \,\&\, \psi)$ *which give.*

LEMMA 1.8. $p\|\vdash \sim\sim(\phi \to \psi)$ *if and only* $p\|\vdash \phi \to \psi$.

That is not a property of for instance the forcing relation defined in |4| where both & and V are taken for basic logic symbols and a part of the definition of a forcing relation is

$p\|\vdash \phi \vee \psi$ if and only if either $p\|\vdash \phi$ or $p\|\vdash \psi$

while we have

LEMMA 1.9. *There exists* $q \geq p$, $q\|\vdash \phi \vee \psi$ *if and only if there exists* $q \geq p$, $q\|\vdash \phi$ *or* $q\|\vdash \psi$.

Secondly, the result of Theorem 1.6. cannot be generalized i.e. an analogous assertion for infinitary logics with equality does not hold even if we kept "the syntactic apparatus" of the finitary logic possible enriched by $PC_\infty 2$. Namely it is easy to prove

LEMMA 1.10. *For all* $p \in C$, $p\|\vdash \sim\sim PC_\infty 2$ *(that is p weakly forces any sentence belonging to the axiom scheme* $PC_\infty 2$ *)*

but let us try to check $p\|\vdash \sim\sim\forall v\forall u(v = u \to (\phi \to \phi'))$ where ϕ is a formula in which u is free for v, ϕ' is the result o

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substituting some (not necessarily all) free occurences of v by u and ϕ is of the form $\Lambda\Psi$ (infinite conjuction). In case of a finitary logic the proof, based on Definition 1.1, is given by induction on the complexity of ϕ.

First of all let us note that (for any ϕ)

$p\|\!\vdash \sim\sim\forall v \forall u\,(v = u \to (\phi \to \phi'))$ if and only if for all closed terms t_1, t_2 $p\|\!\vdash \sim(t_1 = t_2 \,\&\, \phi(t_1) \,\&\, \sim\phi'(t_1, t_2))$.

Let us suppose there exists a condition $q \geq p$ such that $q\|\!\vdash t_1 = t_2 \,\&\, \Lambda\Psi \,\&\, \sim\Lambda\Psi'$. By inductive hypothesis, from $q\|\!\vdash t_1 = t_2$ and $q\|\!\vdash \psi$, $\psi \in \Psi$ follows $q\|\!\not\vdash \sim\psi'$. Hence $q \|\!\not\vdash \sim\Lambda\sim\sim\Psi'$ for in the opposite case for any $r \geq q$ there would exist a formula $\psi' \in \Psi'$ such that $q \|\!\not\vdash \sim\sim\psi'$. Let us fix such a pair r, ψ'. Then for some $s \geq r$ $s\|\!\vdash \sim\psi'$ while also $s\|\!\vdash t_1 = t_2$ and $s\|\!\vdash \psi$, contrary to the inductive assumption. Now the question is, whether $q \|\!\not\vdash \sim\Lambda\sim\sim\Psi'$ is in contradiction with $q\|\!\vdash \sim\Lambda\Psi'$. And the answer is that there is no a general answer, that is that the answer does not follow from the very definition of a forcing relation, as we shall soon see.

In case we wish to get an analogy of Theorem 1.6. for any infinitary logic, two of the possible ways to accomplish this are either to add some new assumptions to the set of conditions or to redefine the set $T^{C}|p|$ in the suitable way.

So if we have the $L_{k\omega}$ logic $(k > \omega)$ it holds (we assume the axiom of choice)

LEMMA 1.11. *If a partially ordered set of conditions* $\langle C, \leq \rangle$ *has the property that for each* $\lambda < k$ *any nondecreasing sequence* $p_0 \leq p_1 \leq \ldots \leq p_\alpha \leq \ldots, \alpha < \lambda$ *in* C *has an upper bound then: for each condition* p *(and any set of sentences* Φ, $|\Phi| < k$*)*

(*) $p\|\!\vdash \Lambda\sim\sim\Phi$ *if and only if* $p\|\!\vdash \sim\sim\Lambda\Phi$

P r o o f. We shall consider only the less trivial implication. Let $\Phi = \{\phi_\gamma \mid \gamma < \lambda (< k)\}$, $p, q \in C$ and $p\|\!\vdash \Lambda\sim\sim\Phi$ and let us suppose that for each $\alpha < \lambda$ we have already constructed a sequence $q \leq p_0 \leq \ldots \leq p_\beta \leq \ldots, \beta < \alpha$ so that $p_\beta\|\!\vdash \phi_\gamma$ for each $\gamma \leq \beta$.

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If α is a limit ordinal just the assumption on partially ordered set enables us to extend the sequence with a new condition p_α ($p_\alpha \geq p_\beta$ for $\beta < \alpha$) which forces all formulas ϕ_γ, $\gamma \leq \alpha$ (in other words we can simply "bridge the gap " between successor and limit ordinals). The case that α is a successor ordinal is clear (compare with 1.5 (c)).

In the end, let us note only that the other (more trivial) implication always holds.

From (*) follows directly $p \Vdash \sim\sim \forall v \forall u(v = u \to (\phi \to \phi'))$ (thus, in general, in the notation from |1| $p \Vdash \sim\sim$ PC 11) also for infinitary logics because of

LEMMA 1.12. *The conditions "(*)" and "for each* p $p \Vdash \sim \bigwedge \phi$ *if and only if* $p \Vdash \sim \bigwedge \sim\sim \phi$" *are equivalent.*

P r o o f. An immediate consequence of Definition 1.1

However the condition on a partially ordered set from the previous lemma is not necessary in order that (*) holds. The following example shows this.

EXAMPLE 1.13. *Let* M *be an infinitely countable model of a countable language and let* Δ *be its diagram (the set of all atomic and negations of atomic sentences of the language* $L(M)$ *which hold in the model* M_M*). Let us enumerate the sentences of* $\Delta = \Phi_1 \cup \Phi_2$ *where* Φ_1 *and* Φ_2 *are, respectively, the set of atomic, that is negations of atomic sentences from* Δ *in such a way that* $\Phi_1 = \{\phi_n \mid n \in \omega\}$ *and* $\Phi_2 = \{\phi_{\omega+k} \mid k \in \omega\}$ *Further let* $C = \{p_\alpha \mid \alpha < \omega+\omega\}$ *where* $p_\alpha = \{\phi_\beta \mid \beta \leq \alpha\}$*, be partially ordered by the inclusion relation and let us determine a forcing relation on* $C \times \mathrm{SENT}(L(M)_{\omega_1\omega})$ *by:*

for atomic ϕ $p_\alpha \Vdash \phi$ *if and inly if* $\phi \in p_\alpha$

For the forcing system $\langle C, \Vdash, L(M)_{\omega_1\omega}\rangle$ (one can easily check that the given triple is really a forcing system) (*) holds i.e. for any condition $p = p_\alpha$ and any set Φ of sentences, $|\Phi| \leq \omega$

$p \Vdash \sim\sim \bigwedge \Phi$ if and only if $p \Vdash \bigwedge \sim\sim \Phi$

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Let us prove this. Since $\langle C, \subseteq \rangle$ is linearly ordered (with the least element $p_o = \{\phi_o\}$) for any β, γ $(< \omega + \omega)$ and each sentence ϕ of the language $L(M)_{\omega_1\omega}$

$$p_\beta \Vdash \sim\sim \phi \quad \text{if and only if} \quad p_\gamma \Vdash \sim\sim \phi$$

Hence it is enough to check that

$$p_\omega \Vdash \sim\sim \wedge \Phi \quad \text{if and only if} \quad p_\omega \Vdash \wedge \sim\sim \Phi$$

Namely, on condition the assertion $p_\alpha \Vdash \sim\sim \wedge \Phi$ iff $p_\alpha \Vdash \wedge \sim\sim \Phi$ is true for at least one condition p_α (*) follows easily. For let $\beta \neq \alpha$ and $p_\beta \Vdash \wedge \sim\sim \Phi$. Then by the above $p_\alpha \Vdash \wedge \sim\sim \Phi$ whence $p_\alpha \Vdash \sim\sim \wedge \Phi$ and therefore $p_\beta \Vdash \sim\sim \wedge \Phi$

But for any sentence ϕ

$p_\omega \Vdash \phi$ if and only if $p_\omega \Vdash \sim\sim \phi$ (then clearly $p_\omega \Vdash \wedge\sim\sim\phi$ implies that $p_\omega \Vdash \wedge \Phi$ and consequently $p_\omega \Vdash \sim\sim \wedge \Phi$).

The proof is by induction on the complexity of ϕ .

If ϕ is atomic and $p_\omega \Vdash \sim\sim \phi$ then $\phi \in \Phi_1$ and so $p_\omega \Vdash \phi$ since $\Phi_1 \subseteq p_\omega$.

If ϕ is $\exists v \psi(v)$ and $p_\omega \Vdash \sim\sim \exists v \psi(v)$ then for some $n \in \omega$ and some closed term t $p_{\omega+n} \Vdash \psi(t)$. Thus $p_\omega \Vdash \sim\sim \psi(t)$ and by the inductive hypothesis $p_\omega \Vdash \psi(t)$ i.e. $p_\omega \Vdash \exists v \psi(v)$.

Other cases are even more trivial.

EXAMPLE 1.14. *Let all suppositions, except the enumeration of* Δ *, be as in the previous case. Now we shall suppose that*

$$\Delta = \{\phi_n \mid n \in \omega\} \quad (\textit{thus} \quad C = \{p_n \mid n \in \omega\}, \quad p_n = \{\phi_o, \ldots, \phi_n\}).$$

It is easy to verify that for all $j \in \omega$

$$p_j \Vdash \wedge \sim\sim \Delta \quad \textit{and} \quad p_j \Vdash \sim \wedge \Delta$$

Thus in particular $p_j \not\Vdash \sim\sim \wedge \Delta$ *and we see that in this example of the forcing system (*) does not hold.*

We have only proved that (*) implies $T^C|p| \vdash$ PC 11 (i.e. for any ϕ which belongs to the axiom scheme PC 11 $T^C|p| \vdash \phi$ that is $\phi \in T^C|p|$). Thus in case (*) does not hold we have to check separately whether $T^C|p| \vdash$ PC 11 holds or not.

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The next example, however, confirms our second remark given after 1.9.

EXAMPLE 1.15. *Let* L *be a language containing just equality relation* = *and let* M *be an infinitely countable model for* L *in which* = *is interpreted as an equivalence relation such that at least one equivalence class contains infinitely many elements. Let* A *be such a class and let* a, b *be two elements of* A. *Let* $\Delta = \Delta_1 \cup \Delta_2$ *be the positive diagram of* M, *where* Δ_1 *is the set of atomic sentences in which constant* c_a *appears, corresponding to the element* a ($\Delta_2 = \Delta \setminus \Delta_1$) *and let us enumerate the sentences of* Δ *so that* $\Delta_1 = \{\phi_n | n \in \omega\}$ *and* $\Delta_2 = \{\phi_{\omega+k} | k \in \omega\}$. *Again we put* $p_\alpha = \{\phi_\beta | \beta < \alpha\}$, $C = \{p_\alpha | \alpha < \omega+\omega\}$ *and define for the language* $L(M)_{\omega_1\omega}$ *a forcing relation as before.*

Now there is no one condition which would force weakly $c_a = c_b \to (\wedge \Delta_1 \to \wedge \Delta_1')$ *where* Δ_1' *is a result of the substitution of constant* c_a *by* c_b *in the sentences of* Δ_1, *for*

$$p_\omega \Vdash c_a = c_b \,\&\, \wedge \Delta_1 \,\&\, \sim \wedge \Delta_1'$$

In particular no condition forces weakly $\forall v \forall u (v = u \to (\wedge \Delta_1(v) \to \to \wedge \Delta_1'(u))$ *where* $\Delta_1'(v)$ *and* $\Delta_1'(u)$ *are obtained from* Δ_1 *that is* Δ_1' *by substituting constants* c_a, c_b *by, the variables* v *and* u *respectively.*

In |1| in order to obtain an analogous result to 1.6 for infinitary logics the notion of $T^C|p|$ is redefined. For that purpose firstly, the concept of "weak" formulas is introduced.

DEFINITION 1.16. *For a formula* ϕ *of (infinitary) logic* L *we define a "weak" formula* ϕ^{wk} *as follows:*

(i) *if* ϕ *is atomic* ϕ^{wk} *is* $\sim\sim\phi$;

(ii) *if* ϕ *is* $\wedge\Psi$ ϕ^{wk} *is* $\wedge\Psi^{wk}$ *(this case includes a finite conjuction);*

(iii) *if* ϕ *is* $\exists v \psi(v)$ ϕ^{wk} *is* $\sim\sim \exists v \psi^{wk}(v)$

and (iv) *if* ϕ *is* $\sim\psi$ ϕ^{wk} *is* $\sim\psi^{wk}$.

From the aspect of a forcing relation, as long as finitary logics are considered, nothing new is obtained by "weak"

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formulas because of

LEMMA 1.17. *If* $\langle C, \Vdash, L\rangle$ *is a forcing system where* L *is a finitary language then for any condition* p *and any sentence* ϕ *of* L

$$p \Vdash \sim\sim \phi \quad \textit{and only if} \quad p \Vdash \phi^{wk}$$

But independent of whether a logic L is finite or not it always holds:

LEMMA 1.18. $p \Vdash \sim\sim \phi^{wk}$ *if and only if* $p \Vdash \phi^{wk}$, and hence

$$p \Vdash \bigwedge \sim\sim \Phi^{wk} \textit{ iff } p \Vdash \bigwedge \Phi^{wk} \textit{ iff } p \Vdash \sim\sim \bigwedge \Phi^{wk}.$$

So as we see "weak" formulas enable us to "draw out" the double negation in front of the infinite conjuction, moreover, to eliminate it. Now it follows directly from the consideration made after 1.10 and Lemma 1.12

LEMMA 1.19. $p \Vdash (PC11)^{wk}$.

This lemma also follows from parts (a) and (b) of

LEMMA 1.20. *Let* $\langle C, \Vdash, L_{\lambda\mu}\rangle$ *be a forcing system. Then (for any* $p \in C$ *):*

(a) *if* $p \Vdash (\phi \to \psi)^{wk}$ *and* $p \Vdash \phi^{wk}$ *then* $p \Vdash \psi^{wk}$;

(b) $p \Vdash (\bigwedge\sim\sim\Phi \to \bigwedge\Phi)^{wk}$ *(i.e.* $p \Vdash (PC_{\infty}1)^{wk}$) ;

(c) $p \Vdash \big(\bigwedge_{\phi\in\Phi} (\psi \to \phi) \to (\psi \to \bigwedge\Phi)\big)^{wk}$ *whence:* $p \Vdash (\psi \to \phi)^{wk}$ *for each* $\phi \in \Phi$ *implies* $p \Vdash (\psi \to \bigwedge\Phi)^{wk}$;

(d) *if* $\phi(v_1, \ldots, v_\mu)$, $\mu < \lambda$ *is a quantificational formula for any closed terms* $t_1, \ldots, t_\mu$ $p \Vdash \phi^{wk}(t_1, \ldots, t_\mu)$.

Thus for the generalized notion of $T^C|p|$ (see Lemma 1.17) given by

DEFINITION 1.21. *Let* $\langle C, \Vdash, L\rangle$ *be a forcing system. For* $p \in C$

$$T^C|p| = \{\phi \in SENT \mid p \Vdash \phi^{wk}\}$$

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(again for $T^C|0|$ *we use only* T^C *)*

one obtains

THEOREM 1.22. |1| *Let* $\langle C, ||-, L\rangle$ *be a forcing system where* L *is an infinitary logic with the set of axioms and rules of inference* Λ_o. *Then for any* $p \in C$

(1) $T^C|p|$ *is a consistent deductively closed set ;*

(2) *if* $\phi(v_1,\ldots,v_\mu)$ *is formula of the language* L *and* $\vdash_L \phi(v_1,\ldots,v_\mu)$ *then for any closed terms* $t_1,\ldots,t_\mu$ $\phi(t_1,\ldots,t_n) \in T^C|p|$

The next theorem (together with example 1.14) shows that the introduction of "weak" formulas is necessary while dealing with infinitary logics even in they are not with equality whenever we want to have at disposal either $PC_\infty 1$ or E_∞.

THEOREM 1.23. *The following are equivalent:*

(a) *(*) ;*

(b) *for each condition* p $p||-\sim\sim\phi$ *if and only if* $p||-\phi^{wk}$;

(c) *for each condition* p $p||-\sim\sim PC_\infty 1$

and (d) *for each condition* p *from* $p||-\sim\sim(\psi\to\phi)$ *for all* $\phi\in\Phi$ *follows* $p||-\sim\sim(\psi\to\Lambda\Phi)$.

P r o o f. (a) → (d) Since weak forcing preserves modus ponens (this assertion is a part of Theorem 1.6) (d) is according to Lemma 1.7 equivalent to

(d´) for each condition p $p||-\Lambda\sim\sim(\psi\to\phi)\to(\psi\to\Lambda\Phi))$ $\phi\in\Phi$

i.e. $p||-\sim(\bigwedge_{\phi\in\Phi}\sim(\psi\wedge\sim\phi)\ \&\ \psi\ \&\sim\Lambda\Phi)$

We think it is simpler to prove (a) → (d´).

Let us suppose (a) holds but for some condition p there exists a condition $q\geq p$ such that $q||-\Lambda\sim(\psi\wedge\sim\phi)\ \&\ \psi\ \&\sim\Lambda\Phi$. By 1.12 $\phi\in\Phi$ $q||-\sim\Lambda\sim\sim\Phi$ whence for some $\phi_o\in\Phi$ $q\not||-\sim\sim\phi_o$. Hence for some condition $r\geq q$ $r||-\sim\phi_o$ but then $r|\!\!-\sim(\psi\ \&\sim\phi_o)$ and $r|\!\!-\psi\ \&\sim\phi_o$, a contradiction.

(d) → (c) By 1.5 (c) from $p||-\sim\sim(\Lambda\sim\sim\Phi\to\sim\sim\phi)$ (for any $\phi\in\phi$) (Lemma 1.10) and $p||-\sim\sim(\sim\sim\phi\to\phi)$ follows $p||-\sim\sim((\Lambda\sim\sim\Phi\to\sim\sim\phi)$ $\&\ (\sim\sim\phi\to\phi))$ thus also $p||-\sim\sim(\Lambda\sim\sim\Phi\to\phi)$. Because of (d)

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$p\|\!\vdash \sim\sim(\Lambda\sim\sim\phi \to \Lambda\phi)$ that is $p\|\!\vdash \sim(\Lambda\sim\sim\phi \,\&\, \sim\Lambda\phi)$.

(c) → (b) On the assumption (c) holds we prove (b) by induction on the complexity of formulas. Of course (see 1.17) the only interesting case is when ϕ is of the form $\Lambda\Psi$

$p\|\!\vdash (\Lambda\Psi)^{wk}$ iff for each $\psi \in \Psi$ $p\|\!\vdash \psi^{wk}$ iff (by the inductive hypothesis) for each $\psi \in \Psi$ $p\|\!\vdash \sim\sim\psi$ iff $p\|\!\vdash \Lambda\sim\sim\Psi$ iff $p\|\!\vdash \sim\sim\Lambda\Psi$ (in the last step we use: $p\|\!\vdash \Lambda\sim\sim\Psi$ and (c) imply that for any $q \geq p$ $q\|\!\not\vdash \sim\Lambda\Psi$, consequently $p\|\!\vdash \sim\sim\Lambda\Psi$).

(b) → (a) We have just proved that always $p\|\!\vdash (\Lambda\phi)^{wk}$ if and only if $p\|\!\vdash \Lambda\sim\sim\phi$ and (b) gives us also $p\|\!\vdash (\Lambda\phi)^{wk}$ if and only if $p\|\!\vdash \sim\sim\Lambda\phi$.

After all this it seems natural to put the question of the appropriatness of defining a forcing relation taking for the basic symbol Λ rather than V. For with (see |2|) $p\|\!\vdash V\phi$ if and only if for some $\phi \in \Phi$ $p\|\!\vdash \phi$ we would obtain the desired $p\|\!\vdash \Lambda\phi$ if and only if $p\|\!\vdash \Lambda\sim\sim\Phi$ (where now $\Lambda\Phi$ replaces $\sim V\sim\Phi$) and therefore also

$$p\|\!\vdash \phi^{wk} \quad \text{if and only if} \quad p\|\!\vdash \sim\sim\phi$$

which would make the introduction of "weak" formulas unnecessary. Our justification could be that the presented system of axioms and rules of inferences for infinitary logic is in wide use.

Let us here also note that disregarding the way we have defined the forcing relation in case L is a fragment of the language $L_{k^+\omega}$ of power $\leq k$, where k is a regular cardinal and D (a set of new constants) of cardinality κ a condition on C like that we use in Lemma 1.11 is put in order that the Generic Model Theorem holds (|2|).

§ 2. The following example is taken from |1|

EXAMPLE 2.1. Let L be a logic with the set of axioms and rules of inference Λ_o, T a theory consistent in L, A_1 and A_2 sets, respectively, of new constants that is, new function and relation symbols, where $|A_2| \leq |A_1| = \bigcup_{\phi \in L} |\phi| \leq |L|$ $(= \kappa)$ and

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$F \subseteq L(A_1 \cup A_2)$ a set with the following properties:

(1) $\phi \in F$ implies $sub(\phi) \subseteq F$ ($sub(\phi)$ is the set of all subformulas of ϕ);

(2) $\phi \in F$ implies $\sim\phi \in F$;

(3) if $\Phi \subseteq F$ and $|\Phi| < \kappa_o$, where κ_o is the supremum of the length of proofs in L (thus λ if L is a fragment of logic $L_{\lambda\mu}$- we recall the definition: $T \vdash \phi$ iff there is a subset Δ of T such that $|\Delta| < \kappa_o$ and $\vdash \bigwedge \Delta \rightarrow \phi$) then $\bigwedge\Phi \in F$;

(4) if $\phi(v) \in F$ and $c \in A_1$ then $\phi(c) \in F$;

and (5) for each formula ϕ from F there exists a constant $c \in A_1$ which does not appear in ϕ.

Let C be a set partially ordered by inclusion whose elements are all subsets (p) of $SENT(F) = SENT(L(A_1 \cup A_2)) \cap F$ which satisfy:

(i) if $p \in C$ then $|p| < \kappa_o$

(ii) for $p \in C$ $T \cup p$ is consistent theory in $L(A_1 \cup A_2)$

and (iii) formula belonging to $p \in C$ is not a conjunction.

We define a unary relation $\Vdash$ on $C \times AT(L(A_1 \cup A_2))$ by $p \Vdash \phi$ if and only if $\phi \in p$
and assume that it is extended to a relation on $C \times SENT(L(A_1 \cup A_2))$ resembling the forcing relation.

Our first remark would be that without additional assumptions about the set F the given relation does not have to be a forcing relation Even the assertion:

if t_1, t_2 are closed terms occurring in formulas of F and $\phi(v) \in F$ then

(i) $\emptyset \Vdash \sim\sim (t_1 = t_1)$

and (ii) for each p there exists $q \supseteq p$ such that either $p \nVdash t_1 = t_2$ or $p \nVdash \phi(t_1)$ or $q \Vdash \phi(t_2)$ does not have to hold always.

It seems most natural to introduce the condition that F contains the complete corresponding finitary logic and then if λ is a singular cardinal necessarily to weaken the condition (3) in order that (5) be kept. The condition (3) is too strong

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anyhow. Let us also say that the sets $F\phi_1$, $F\phi_2$ and F_o we use in the proof of the weak interpolation theorem do not necessarily satisfy it. In addition to all that (3) is, together with (4), without an extra, great restriction, in collision with (5) for λ singular.

The next results are interesting in themselves and even if a forcing relation is not considered they will be useful in the application of forcing.

Let λ be a regular cardinal (this condition is only to simplify the "story"), L a fragment of Logic $L_{\lambda\mu}$ (with the set of axioms and rules of inference Λ_o) and A_1, A_2 and T is in 2.1 and as for C we omit condition (iii), which in our opinion, in the given consideration does not play any special role. We define the relation $||\!-$ on $C \times SENT(L(A_1 \cup A_2))$ as in 2.1 with the exception that now we put

$p||\!- \exists v\phi(v)$ if and only if there exists $c \in A_1$ such that $p||\!- \phi(c)$ (the alternative would be the strengthening of (4) by

(4′) if $\phi(v) \in F$ and t is a closed term then $\phi(t) \in F$.)

For relation $||\!-$ holds

LEMMA 2.2. $p||\!- \phi^{wk}$ *if and only if* $p||\!-\sim\sim \phi^{wk}$.

P r o o f. By induction on the complexity of formulas.

LEMMA 2.3. *If* $\phi \in SENT(F)$, $p \in C$ *and (1)* $\phi \in p$; *(2)* $p||\!- \phi^{wk}$; *(3)* $T \cup p||\!\not\sim \phi$ *and (4) there exists* q *such that* $q \supseteq p \cup \{\phi\}$ *then*

(a) (1) → (2); (b) (2) → (3) *and* (c) (3) → (4).

P r o o f. (c) is trivial. We prove (a) and (b) (simultaneously) by induction on the complexity of formulas. In regard to the reformulation made we shall give only the supplement (with necessary correction) of the proof from (1).

Let $\Lambda\Phi \in p$, $q \supseteq p$ and $\phi \in \Phi$: In view of the fact that $T \cup q \cup \{\phi\}$ is consistent $r = q \cup \{\phi\} \in C$ and by the inductive assumption $r||\!- \phi^{wk}$. It follows $p||\!-\sim\sim \phi^{wk}$ i.e. $p||\!- \phi^{wk}$, accordingly

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also $p||\!\!-\ (\Lambda\phi)^{wk}$.

If $p||\!\!-(\Lambda\Phi)^{wk}$ but not $T \cup p||\!\!\not\sim \Lambda\Phi$ then $T \cup p \cup \{\sim\Lambda\Phi\}$ is consistent. Consequently for some $\phi \in \Phi$ $T \cup p \cup \{\sim\Lambda\Phi, \sim\phi\}$ is consistent. But for $q \supseteq p \cup \{\sim\Lambda\Phi, \sim\phi\}$ $q||\!\!-\sim\phi^{wk}$ is in contradiction with $p||\!\!-\phi^{wk}$ and $q||\!\!\not\sim \sim\phi^{wk}$ implies the existence of some $r \supseteq q$ such that $r||\!\!-\phi^{wk}$ and because of it $T \cup r \not\vdash \sim\phi$ while $\sim\phi \in r$ is a contradiction again.

If $\exists v\phi(v) \in p$ then for some $c \in A_1$ $T \cup p \cup \{\phi(c)\}$ is consistent (it is easy to see that this is true for any constant c of A_1 not appearing in sentences of p). Analogously for any $q \supseteq p$ there exists c from A_1 such that $r = q \cup \{\phi(c)\} \in C$. Since by the inductive hypothesis $r||\!\!-\phi^{wk}(c)$ (that is $r||\!\!-\exists v\ \phi^{wk}(v)$) $p||\!\!-\sim\sim \exists v\phi^{wk}$ i.e. $p||\!\!-(\exists v\phi)^{wk}$.

From $p||\!\!-(\exists v\phi(v))^{wk}$ follows the existence of $r \in C$ and $c \in A_1$ such that $r \supseteq p$ and $r||\!\!-\phi^{wk}(c)$. Assumption $T \cup r \not\vdash \sim\phi(c)$ implies $T \cup r \not\vdash \forall v\sim\phi(v)$, therefore $T \cup p \not\vdash \sim\exists v\phi(v)$.

COROLLARY 2.4. *For* $\phi \in SENT(F)$ *and* $p \in C$

$$p||\!\!-\phi^{wk} \quad \textit{if and only if} \quad T \cup p \vdash \phi .$$

REMARK. In regard to the supposition that λ is a regular cardinal the restriction $|T| < \lambda$ is unnecessary. In the opposite case (when λ is singular), we introduce it because of the application of rule E_∞.

The rest of the paper shall be devoted to the proof of the so-called weak interpolation theorem which is to replace, in general, the invalid interpolation theorem for infinitary logics (see |1|, |4|).

THEOREM 2.5. *Let* ϕ_1 *and* ϕ_2 *be two sentences of the given logic* $L_{\lambda\mu}$ *with equality (and the set of axioms and rules of inference* Λ_o*) and let* $\vdash \phi_1 \to \phi_2$ *. Then there exists a sentence* ϕ *of logic* $L_{\lambda\mu}$ *such that* $\vdash \phi_1 \to \phi$ *and* $\vdash \phi \to \phi_2$ *and that each constant and each function and relation symbol with the exception of* $=$*, which occurs in* ϕ*, occurs as well in both* ϕ_1 *and* ϕ_2.

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The main idea of the proof, which to a great extent suggests technical solutions, is taken on from the proof of the Interpolation Theorem in (classical) $L_{\omega\omega}$ logic where under the assumption that there is no interpolant between ϕ_1 and ϕ_2 is being shown the existence of a Hintikka theory containing ϕ_1 and $\sim\phi_2$, hence the existence of a model (corresponding canonical model) for ϕ_1 and $\sim\phi_2$.

Now in accordance with replacing $\models$ with $\vdash$ we use syntactical apparatus. For that purpose we shall extend, firstly, the language L by a set of (new) constants A of cardinality $\bigcup_{\xi\in L}|\xi|$ and then in $L(A)_{\lambda\mu}$ define set $F\phi_i$, i=1,2 as the set of all formulas with the property that each constant and each function and relation symbol (different from =) of the language L, which occurs in them occurs in ϕ_i and which contain no more constants from A than it is permissible to have quantifiers (which should enable us to "eliminate", if necessary, these constants from the relevant formula). In general $F\phi_i$ satisfies all but the third item of the definition from 2.1. However if $\Phi \subseteq F\phi_i$ and $|\Phi| < cf\mu$ then $\wedge\Phi \in F\phi_i$. Thus, and by analogy with the proof of Craig's theorem, we are taking for elements of C all the subsets, including the empty set, $P = P_1 \cup P_2$ of $SENT(F\phi_1 \cup F\phi_2)$ (we assume that $P_i \subset F\phi_i$, i=1,2) of the cardinality less than $cf\mu$ and such that the union of theories $Thm(\wedge P_i) = \{\xi \in F_o = F\phi_1 \cap F\phi_2 | \wedge P_i \vdash \xi\}$, i=1,2 is consistent. The relation $\| \subseteq C \times SENT(L(A)_{\lambda\mu})$ is defined like the relation $\|\!\sim$ in 2.3. This time using the sign $\|$ instead of $\|\!\vdash$ we emphasize that $\|$ is not necessarily a forcing relation.

To the proofs of the following several lemmas which can be found in |1| should be added, because it follows from the reasons unmentioned there, the real possibility of the assumption that sets Δ, applied in them, belong to F_o.

LEMMA 2.6. *If* $\xi_i \in F\phi_i$, i=1,2 *and* $\xi \in F_o$ *then* $Thm(\xi_1 \& \xi) \cup Thm(\xi_2)$ *is consistent in and only if* $Thm(\xi_1) \cup Thm(\xi_2 \& \xi)$ *is consistent.*

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LEMMA 2.7. *If* $\xi_i \in F\phi_i$, i=1,2 *and* $\mathrm{Thm}(\xi_1) \cup \mathrm{Thm}(\xi_2)$ *is inconsistent there exists an interpolant* ξ_o *between* ξ_1 *and* $\sim\xi_2$ ($\vdash \xi_1 \to \xi_o$ *and* $\vdash \xi_o \to \sim\xi_2$).

LEMMA 2.8. *If* $\xi_i \in F\phi_i$, i=1,2 *and* $\mathrm{Thm}(\xi_1) \cup \mathrm{Thm}(\xi_2)$ *is consistent then* $\xi_1 \cup \mathrm{Thm}(\xi_1) \cup \mathrm{Thm}(\xi_2)$ *is consistent too.*

LEMMA 2.9. *If* $\xi_i \in F\phi_i$, i=1,2, $\phi \in F\phi_1$ *and* $\mathrm{Thm}(\xi_1 \,\&\, \phi) \cup \cup \mathrm{Thm}(\xi_2)$ *is inconsistent then* $\{\xi_1\} \cup \mathrm{Thm}(\xi_2) \vdash \sim\phi$ *and* $\mathrm{Thm}(\xi_1 \,\&\, \&\sim\phi) \cup \mathrm{Thm}(\xi_2)$ *is consistent.*

LEMMA 2.10. *If* $\xi_i \in F\phi_i$, i=1,2, $\Lambda\Phi \in F\phi_1$ *and* $\mathrm{Thm}(\xi_1 \,\&\, \sim\Lambda\Phi) \cup \mathrm{Thm}(\xi_2)$ *is a consistent theory then for some* $\phi \in \Phi$ $\mathrm{Thm}(\xi_1 \&\sim\Lambda\Phi\&\sim\phi) \cup \mathrm{Thm}(\xi_2)$ *is also consistent.*

LEMMA 2.11. *Let* $P \in C$ *and* $\psi \in F\phi_i$ ($i \in \{1,2\}$). *Then* (1) $\to$ (2) *and* (2) $\to$ (3) *where* (1) $\psi \in P$; (2) $P \Vdash \psi^{wk}$ *and* (3) *there exists* Q *in* C *such that* $P \cup \{\psi\} \subseteq Q$.

P r o o f. By induction on the complexity of the formula ϕ. Since lemmas 2.6 - 2.10 have already been given there is little more left to be done. The case ψ is $\exists v\phi(v)$ shall serve as an example (the other cases, we think, are easier).

Let $\exists v\phi(v) \in F\phi_1$ and $\exists v\phi(v) \in P_1 \subseteq P_1 \cup P_2 = P$ (this is, of course, no restriction at all). Then for some $c \in A$ $\mathrm{Thm}(\Lambda P_1 \,\&\, \&\, \phi(c)) \cup \mathrm{Thm}(\Lambda P_2)$ is consistent for in the opposite case $\Lambda P_1 \cup \mathrm{Thm}(\Lambda P_2) \vdash \sim\phi(c)$ for each $c \in A$ (2.9) but if c is a constant not occurring in either ΛP_1 or ΛP_2 we would obtain $\Lambda P_1 \cup \mathrm{Thm}(\Lambda P_2) \vdash \sim \exists v\phi(v)$, contradictory to 2.8 (2.6 makes consideration of theory $\mathrm{Thm}(\Lambda P_1) \cup \mathrm{Thm}(\Lambda P_2 \& \phi(c))$ in case $\exists v\phi(v) \in F_o$ superfluous. Therefore for all $Q \in C$, $Q \supseteq P$ there exists c in A so that $Q \cup \{\phi(c)\} = R \in C$ whence because of $R \Vdash \phi^{wk}(c)$ $P \Vdash \sim\sim \exists v\phi^{wk}(v)$, accordingly $P \Vdash (\exists v\phi(v))^{wk}$.

On condition $P \Vdash (\exists v\phi(v))^{wk}$ $Q \Vdash \phi^{wk}(c)$ for some $Q \supseteq P$ and some $c \in A$. Then $Q \cup \{\psi(c)\} \subseteq R \in C$ and so $R \cup \{\exists v\phi(v)\} \in C$ also.

However, what else we need is a forcing relation ($\Vdash$) such that a set of conditions is C and for $\phi \in F\phi_i$, ($i \in \{1,2\}$) (and $P \in C$) $P \Vdash \phi^{wk}$ if (and only if) $P \Vdash \phi^{wk}$.

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For with it here is the proof. Namely, according to 2.7 it is sufficient to prove $\{\phi_1, \sim\phi_2\} \notin C$. But the assumption $P = \{\phi_1, \sim\phi_2\} \in C$ leads to a contradiction because of theorem 1.22 (and hypothesis $\vdash \phi_1 \to \phi_2$) $P \Vdash \sim(\phi_1 \& \sim\phi_2)^{wk}$ while according to 2.11 $P \Vdash \phi_1^{wk} \& \sim\phi_2^{wk}$, thus also $P \Vdash \phi_1^{wk} \& \sim\phi_2^{wk}$.

In the following we assume that language L contains of the nonlogical symbols only those occurring in formulas ϕ_1 and ϕ_2 and if it is not already included, the relation symbol $=$, that is $L = L_1 \cup L_2 \cup \{=\}$ where L_1 and L_2 are languages, elements of which are symbols from ϕ_1 and ϕ_2, respectively.

For a closed term t of the language L(A) we will say that it is basic if either t is a constant or t is of the form $f(c_1, \ldots, c_n)$ where f is an n-ary function symbol and $c_1, \ldots, c_n$ are elements of A |5|.

LEMMA 2.12. *For each basic term* t *and each* $P \in C$ *there exists* $c \in A$ *and* $Q \in C$ *such that* $P \cup \{t = c\} \subseteq Q$.

P r o o f. An immediate consequence of lemmas 2.8, 2.9.

Let t be a closed term, $P \in C$ and $c \in A$. We define relation $P \Vdash t = c$ in the following way:

for $t = c \in F\phi_1 \cup F\phi_2$ by : (a) $P \Vdash t = c$ if and only of $t = c \in P$;

otherwise, inductively (on the complexity of the term t) and according to (a): if $t = f(t_1, \ldots, t_n)$ (and $f(t_1, \ldots, t_n) = c \notin F\phi_1 \cup F\phi_2$) $P \Vdash t = c$ if and only if there exist elements $c_1, \ldots, c_n$ from A such that $P \Vdash t_i = c_i$, $i = 1, \ldots, n$ and $P \Vdash f(c_1, \ldots, c_n) = c$.

Relation $P \Vdash c = t$ is analogously determined.

From now on we shall not always accent that the first components (P,Q,R,...) of relation $\Vdash$ are elements of C (i.e. we shall not permanently repeat $P \in C$, $Q \in C$, $R \in C$, ...). Besides that being most of the proofs of the subsequent lemmas rather tedious than difficult we shall usually omit them.

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LEMMA 2.13. *If* $P||\!\!-\, t=c$ $(P||\!\!-\, c=t)$ *and* $Q \supseteq P$ *then* $Q||\!\!-\, t=c$ $(Q||\!\!-\, c=t)$.

LEMMA 2.14. *If* $P||\!\!-\, t=c$ $(P||\!\!-\, c=t)$ *there exists* $Q, Q \supseteq P$ *and* $Q||\!\!-\, c=t$ $(Q||\!\!-\, t=c)$.

LEMMA 2.15. *For each closed term* t *and any* P *there exist* Q *and* $c \in A$ *such that* $P \subseteq Q$ *and* $Q||\!\!-\, t=c$ $(Q||\!\!-\, c=t)$.

P r o o f. If all the symbols of t are from L_1 (L_2) the assertion follows from the fact that there exists a constant $c \in A$ not occurring in the sentences of P. Otherwise we use the induction on the complexity of t.

LEMMA 2.16. *Let* t_1, t_2 *be closed terms and* $c,d \in A$.

(a) *If* $P||\!\!-\, t_1=c$ *and* $P||\!\!-\, t_1=d$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q||\!\!-\, c=d$.

(b) *If* $P||\!\!-\, t_1=c$ *and* $P||\!\!-\, c=d$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q||\!\!-\, t_1=d$

(c) *If* $P||\!\!-\, t_1=t_2$ *and* $P||\!\!-\, t_1=c$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q||\!\!-\, t_2=c$.

(d) *If* $P||\!\!-\, t_1=t_2$, $P||\!\!-\, t_1=c$ *and* $P||\!\!-\, t_2=d$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q||\!\!-\, c=d$.

Now in the natural way we extend the relation $||\!\!-$ with the remark that we shall use the same symbol for extensions.

DEFINITION 2.17. *Let* $\phi \equiv \rho(t_1,\ldots,t_n)$ *(for* $n=2$ ρ *can be* $=$ *as well) be an atomic sentence (of the language* $L(A)$*) and* $P \in C$. *Relation* $||\!\!-\, \subseteq C \times AT(L(A))$ *is given by*

(a) $P||\!\!-\, \phi$ *if and only if* $\phi \in P$ *for* $\phi \in F\phi_1 \cup F\phi_2$, *that is*

(b) $P||\!\!-\, \phi$ *if and only if there exist constants* $c_1,\ldots,c_n$ *from* A *such that* $P||\!\!-\, t_i=c_i$, $i=1,\ldots,n$ *and* $\rho(c_1,\ldots,c_n) \in P$ *(i.e. according to (a)* $P||\!\!-\, \rho(c_1,\ldots,c_n)$*) if* $\phi \notin F\phi_1 \cup F\phi_2$

(Clearly, requirements $P||\!\!-\, c_i=t_i$ *instead of* $P||\!\!-\, t_i=c_i$, $i=1,\ldots,n$ *would change nothing essentially - 2.14).*

LEMMA 2.18. *If* ϕ *is an atomic sentence,* $P||\!\!-\, \phi$ *and* $Q \supseteq P$ *then also* $Q||\!\!-\, \phi$.

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LEMMA 2.19. *For all closed terms* t_1, t_2, t_3 *and each* $P \in C$ *holds:*

(a) *There exists* $Q, Q \supseteq P$ *and* $Q \Vdash t_1 = t_1$;

(b) *if* $P \Vdash t_1 = t_2$ *there exists* $Q, Q \supseteq P$ *and* $Q \Vdash t_2 = t_1$;

(c) *if* $P \Vdash t_1 = t_2$ *and* $P \Vdash t_2 = t_3$ *there exists* $Q, Q \supseteq P$ *and* $Q \Vdash t_1 = t_3$.

LEMMA 2.20. *Let* $t_i = t_i'$, $i=1,\ldots,n$ *be closed terms,* f *and* ρ *a function that is a relation symbol of the length* n *and* $P \in C$. *If* $P \Vdash t_i = t_i'$, $i=1,\ldots,n$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q \Vdash f(t_1,\ldots,t_n) = f(t_1',\ldots,t_n')$. *If still* $P \Vdash \rho(t_1,\ldots,t_n)$ *there exists* $R, R \supseteq P$ *and* $R \Vdash \rho(t_1',\ldots,t_n')$.

LEMMA 2.21. *Let* t_1, t_2 *and* σ *be closed terms,* σ' *a term obtained by substitution in* σ *(not necessarily all) occurences of* t_1 *by* t_2 *and let* $P \in C$. *If* $P \Vdash t_1 = t_2$ *there exists* Q *such that* $Q \supseteq P$ *and* $Q \Vdash \sigma = \sigma'$.

P r o o f. By induction on the complexity of σ using the previous lemmas.

LEMMA 2.22. *Let* t_1, t_2 *be closed terms and* $\phi(v)$ *an atomic formula (with at most one free variable). Then for each* P *there exists* $Q \supseteq P$ *such that either* $P \Vdash / t_1 = t_2$ *or* $P \Vdash / \phi(t_1)$ *or* $Q \Vdash \phi(t_2)$.

According to 2.18 and 2.22 the relation $\Vdash$ determines a forcing relation (which we denote also by $\Vdash$ and) for which holds:

LEMMA 2.23. *Let* P *be a condition (now in regard to the accepted terminology we again call elements of* C *conditions),* t *a closed term and* $\phi(v)$ *a formula (of the language* L(A)*) with at most one free variable,* c *an element of* A *and let* $P \Vdash t = c$. *Then* $P \Vdash (\phi(t))^{wk}$ *if and only if* $P \Vdash (\phi(c))^{wk}$.

P r o o f. By induction on the complexity of formula ϕ. If $\phi(v)$ is an atomic formula the statement is a direct consequence of the previous three lemmas. Other cases are trivial.

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On the basis of what has been said the proof of (the Weak Interpolation Theorem) 2.5 follows from

THEOREM 2.24. *For* $\phi \in F\phi_1 \cup F\phi_2$ *and* $P \in C$ $P \| \phi^{wk}$ *if and only if* $P \| \vdash \phi^{wk}$.

P r o o f. By induction on the complexity of ϕ. The only interesting case we have is when ϕ is of the form $\exists v\psi(v)$ (where $\psi(v)$ is a formula with at most one free variable).

Let $P \| \vdash (\exists v\psi(v))^{wk}$ (i.e. $P \| \vdash \sim\sim \exists v\psi^{wk}(v)$) and $Q \supseteq P$. Then for some $R \supseteq Q$ and some closed term t $R \| \vdash \psi^{wk}(t)$. By 2.15 there exists $S (\in C)$ and $c \in A$ such that $S \supseteq R$ and $S \| \vdash t = c$. Thus also (2.23) $S \| \vdash \psi^{wk}(c)$ whence $P \| (\exists v\psi(v))^{wk}$ too. The proof in the opposite direction is trivial.

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REZIME

PRIMEDBA O FORSINGU I SLABOJ INTERPOLACIONOJ TEOREMI ZA BESKONAČNE LOGIKE

Naša razmatranja odnose se na rezultate prva tri poglavlja iz |1|. Cilj nam je da bolje osetimo ulogu slabih formula i korigujemo dokaz slabe interpolacione teorere za beskonačne logike.

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U |1| je već nagoveštено da (direktni) dokazi tvrdjenja kao što su: za svako p $p||\!\!-\sim PC_{\omega}1$, ili: pravilo izvodjenja E_{ω} ostaje očuvano, "izgleda da zahtevaju" sledeće svojstvo skupa uslova C: (*) za svako p iz $p||\!\!-\Lambda \sim \Phi$ sledi $p||\!\!-\sim \Lambda \Phi$ (implikacija u suprotnom smeru je uvek tačna). Mi pokazujemo da su ovi iskazi ekvivalentni (dakle i medjusobno ekvivalentni) sa: za svako p $p||\!\!-\sim\sim\phi$ ako i samo ako $p||\!\!-\phi^{wk}$ (Teorema 1.23.). Dovoljan uslov, samo pretpostavljen u |1|, da ova tvrdjenja i važe u slučaju kada posmatramo fragment logike $L_{k\lambda}$, je da svaki nerastući niz uslova dužine $\lambda < k$ ima gornje ograničenje (Lema 1.11). To, medjutim, nije i potreban uslov (Primer 1.13). Iz (*) takodje proizilazi: za svako p $p||\!\!-\sim\sim PC11$, što inače nije u opštem ispunjeno za beskonačne logike (Primer 1.15).

Što se tiče dokaza slabe interpolacione teoreme za beskonačne logike (semantičko $\models$ je zamenjeno sintaktičkim $\vdash$) osnovna zamerka nam je da relacija (||) koja se u njemu koristi ne mora da bude i u slučajevima od stvarnog interesa i nije, forsing relacija dok se u isto vreme koriste osobine forsing relacije (posebno 1.22). Iskrsli problem rešavamo konstrukcijom forsing relacije koja sa datom ima presek "po meri" (Leme 2.12 - 2.23, Teorema 2.24). Učinjene su i neke druge korekcije i poboljšanja.

Komentari uz pojedine stavove treba da doprinesu boljem sagledavanju izložene materije.

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A GENERALIZATION OF THE BOCHNER AND CONSTANT BOCHNER CURVATURE TENSOR

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ABSTRACT

Using the curvature tensor of the complex conformal connection, the tensor (2.28) is constructed. This tensor generalizes the Bochner curvature tensor as well as the complex conharmonic curvature tensor.

Using the curvature tensor of the contact conformal connection, the tensor (5.6) is constructed. This tensor generalizes the contact Bochner curvature tensor as well as the contact conharmonic curvature tensor.

INTRODUCTION

Let (M,g,F) be the Kähler manifold, i.e. an n-dimensional $(n = 2m;\ n \geq 4)$ differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods (U,x^i) in which there are given a tensor field F_j^i and a Riemannian metric g_{ij} satisfying

$$F_j^i F_i^k = -\delta_j^k, \quad g_{ab}F_i^a F_j^b = g_{ij}, \quad \nabla_j F_i^k = 0,$$

where ∇_j is the operator of the covariant differentiation with respect to the Christoffel symbols $\{ {k \atop ji} \}$ formed with g_{ij}.

If we put

$$F_{ji} = F_j^a g_{ai},$$

we have

$$F_{ij} = -F_{ji}, \qquad \nabla_k F_{ij} = 0.$$

Trying to find a complex analogue of the conformal change of the Riemannian connection, K.Yano |1| considered a conformal change of the Hermitian metric:

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(1.1) $\bar{g}_{ij} = e^{2\sigma} g_{ij}$, $\bar{F}^i_j = F^i_j$, $\bar{F}_{ij} = e^{2\sigma} F_{ij}$

where σ is a scalar function, and he looked for an affine connection Γ^i_{jk} such that

$$D_k \bar{g}_{ij} = 0,$$

where D_k denotes the operator of the covariant differentation with respect to Γ^i_{jk}, and such that the torsion tensor S^k_{ji} is given by

$$S^k_{ji} = - F_{ji} \kappa^k ,$$

where κ^k are the components of a vector field. This complex conformal connection is given by

(1.2) $$\Gamma^k_{ji} = \{ {k \atop ji} \} + \delta^k_j \sigma_i + \delta^k_i \sigma_j - g_{ji} \sigma^k + F^k_j \kappa_i + F^k_i \kappa_j - F_{ji} \kappa^k ,$$

were

(1.3) $$\sigma_i = \partial_i \sigma, \quad \sigma^i = \sigma_a g^{ai}, \quad \kappa_i = -\sigma_a F^a_i, \quad \kappa^k = \kappa_a g^{ak} .$$

In the same paper K.Yano proved the Theorem:

If, in a real n-dimensional Kähler manifold $(n \geq 4)$, there exists a scalar function σ such that the complex conformal connection (1.2) is of zero curvature, then the Bochner curvature tensor:

$$B_{kji}{}^h = K_{kji}{}^h + \frac{1}{n+4} (\delta^h_j K_{ki} - \delta^h_k K_{ji} - g_{ji} K_k{}^h + g_{ki} K_j{}^h$$

(1.4.) $$+ F^h_j K^*_{ki} - F^h_k K^*_{ji} - F_{ji} K^*_k{}^h + F_{ki} K^{*h}_j + 2K^*_{kj} F^h_i +$$
$$+ 2K^*_i{}^h F_{kj}) - \frac{K}{(n+2)(n+4)} (\delta^h_j g_{ki} - \delta^h_k g_{ji} +$$
$$+ F^h_j F_{ki} - F^h_k F_{ji} + 2F_{kj} F^h_i)$$

where $K_{kji}{}^h$ is the curvature tensor of $\{ {k \atop ji} \}$ and

$K_{ji} = K_{aji}{}^h$, $K = K_{ij} g^{ij}$, $K^*_{ji} = - K_{ja} F^a_i$, $K^*_j{}^h = K^*_j{}^a g^{ah}$

In paper [2], K.Yano found the contact analogue of the above.

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Namely, let (M,ϕ,ξ,η,g) be the Sasakian manifold, i.e. a $(2m+1)$-dimensional differentiable manifold of class C^∞ covered by a system of coordinate neighborhoods (U,x^i) in which there are given a tensor field ϕ_i^h, a vector field ξ^h, a 1-form η_i and a Riemannian metric g_{ij} satisfying

$$\phi_j^i\phi_i^h = -\delta_j^h + \eta_j\xi^h, \quad \phi_i^h\xi^i = 0, \quad \eta_i\phi_j^i = 0, \quad \eta_i\xi^i = 1,$$

$$g_{ab}\phi_j^a\phi_i^b = g_{ji} - \eta_j\eta_i, \quad \eta_i = g_{ih}\xi^h,$$

$$\phi_{ji} \equiv \phi_j^a g_{ai} = \frac{1}{2}(\partial_j\eta_i - \partial_i\eta_j)$$

$$\nabla_i\eta^h = \phi_i^h, \quad \nabla_j\phi_i^h = -g_{ji}\eta^h + \delta_j^h\eta_i .$$

Then K.Yano found a contact conformal connection as follows [2]:

$$\Gamma_{ji}^h = \left\{ {h \atop ji} \right\} + (\delta_j^h - \eta_j\eta^h)\sigma_i + (\delta_i^h - \eta_i\eta^h)\sigma_j$$
$$- (g_{ji} - \eta_j\eta_i)\sigma^h + \phi_j^h(\kappa_i - \sigma_i) + \phi_j^h(\kappa_j - \sigma_j) - \phi_{ji}(\kappa^h - \sigma^h), \tag{1.5}$$

where

$$\sigma_i = \partial_i\sigma, \quad \sigma^h = \sigma_a g^{ah}, \quad \kappa_i = -\sigma_a\phi_i^a, \quad \kappa^h = \kappa_a g^{ah} .$$

This connection satisfies

$$\mathcal{D}_k(e^{2\sigma}g_{ij}) = 2e^{2\sigma}\sigma_k\eta_i\eta_j, \quad \mathcal{D}_k\phi_j^i = 0, \quad \mathcal{D}_k\eta^h = 0,$$

i.e.

$$\mathcal{D}_k[e^{2\sigma}(g_{ij} - \eta_i\eta_j)] = 0.$$

where $\mathcal{D}_k$ is the operator of the covariant differentation with respect to (1.5), and the function σ satisfies $\sigma_i\eta^i = 0$.

Also, K.Yano proved the Theorem [2]:

If, in a $(2m+1)$-dimensional Sasakian manifold $(2m+1 > 3)$, there exists a scalar function σ such that the contact conformal connection (1.5) is of zero curvature, then the contact Bochner curvature tensor:

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$$B_{kji}{}^{h} = K_{kji}{}^{h}$$

$$-(\delta_k^h - \eta_k\eta^h)\left(\frac{K_{ji}}{2(m+2)} + \frac{6m+8-K}{8(m+1)(m+2)}g_{ji} + \frac{10m+8+K}{8(m+1)(m+2)}\eta_i\eta_j\right)$$

$$+(\delta_j^h - \eta_j\eta^h)\left(\frac{K_{ki}}{2(m+2)} + \frac{6m+8-K}{8(m+1)(m+2)}g_{ki} + \frac{10m+8+K}{8(m+1)(m+2)}\eta_i\eta_k\right)$$

$$-(g_{ji} - \eta_j\eta_i)\left(\frac{K_k^h}{2(m+2)} + \frac{6m+8-K}{8(m+1)(m+2)}\delta_k^i + \frac{10m+8+K}{8(m+1)(m+2)}\eta_k\eta^h\right)$$

(1.6)
$$+(g_{ki} - \eta_k\eta_i)\left(\frac{K_j^h}{2(m+2)} + \frac{6m+8-K}{8(m+1)(m+2)}\delta_j^k + \frac{10m+8+K}{8(m+1)(m+2)}\eta_j\eta^h\right)$$

$$+\phi_k^h\left(\frac{K_{ja}\phi_i^a}{2(m+2)} - \frac{6m+8-K}{8(m+1)(m+2)}\phi_{ji}\right) - \phi_j^h\left(\frac{K_{ka}\phi_i^a}{2(m+2)} - \frac{6m+8-K}{8(m+1)(m+2)}\phi_{ki}\right)$$

$$+\phi_{ji}\left(\frac{K_{ka}\phi^{ha}}{2(m+1)} - \frac{6m+8-K}{8(m+1)(m+2)}\phi_k^h\right) - \phi_{ki}\left(\frac{K_{ja}\phi^{ha}}{2(m+2)} - \frac{6m+8-K}{8(m+1)(m+2)}\phi_j^h\right)$$

$$-\frac{1}{m+2}\left(K_{ia}\phi^{ha}\phi_{kj} + K_{ka}\phi_j^a\phi_i^h\right) + \left(\frac{3m+4}{(m+1)(m+2)} - \frac{K}{2(m+1)(m+2)}\right)\phi_{kj}\phi_i^h$$

$$+\phi_k^h\phi_{ji} - \phi_j^h\phi_{ki} - 2\phi_{kj}\phi_i^h$$

vanishes.

In the present paper we shall determine the tensors generalizing the Bochner curvature tensor and the contact Bochner curvature tensor, respectively. Namely, in section 2, we shall consider the Kähler manifold where there exists a scalar function σ such that

$$\nabla_p\sigma^p + a\sigma_p\sigma^p = 0, \quad a = \text{const}; \quad a \neq \frac{n^2-4}{2(n+1)} . \tag{1.7}$$

Then, using the curvature tensor of the complex conformal connection (1.2) we construct the tensor $H_{kji}{}^{h}$ which does not depend on the function σ. In the case $a = n+2$, $H_{kji}{}^{h}$ reduce to the Bochner curvature tensor. In section 3 we shall give some theorems conserning the tensor $H_{kji}{}^{h}$. In section 4 we shall find

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the complex conharmonic curvature tensor which is another special case of $H_{kji}^{\ \ h}$.

In section 5 we consider (2m+1)-dimensional Sasakian manifold where there exists a scalar function σ satisfying (1.7) such that $a \neq \frac{2(m^2-1)}{2m+1}$. Using the curvature tensor of the contact conformal connection (1.5) we construct the tensor $H_{kji}^{\ \ h}$ generalizing the contact Bochner curvature tensor as well as the conharmonic curvature tensor.

2. GENERALIZATION OF THE BOCHNER CURVATURE TENSOR

We compute the curvature tensor

$$R_{kji}^{\ \ \ h} = \partial_k \Gamma_{ji}^{\ \ h} - \partial_j \Gamma_{ki}^{h} + \Gamma_{kt}^{h}\,\Gamma_{ji}^{t} - \Gamma_{jt}^{h}\Gamma_{ki}^{t}$$

of the complex conformal connection (1.2) and we follow the way indicated in |1|. We find

$$R_{kji}^{\ \ \ h} = K_{kji}^{\ \ \ h} + \delta_j^h p_{ki} - \delta_k^h p_{ji} - p_k^{\ h} g_{ji} + p_j^{\ h} g_{ki}$$
$$+ F_j^h q_{ki} - F_k^h q_{ji} - q_k^{\ h} F_{ji} + q_j^{\ h} F_{ki} - \alpha_{kj} F_i^h - \beta_i^{\ h} F_{kj}\,, \tag{2.1}$$

where

$$p_{ji} = \nabla_j \sigma_i - \sigma_i \sigma_j + \kappa_i \kappa_j + \frac{1}{2}\,\sigma_a \sigma^a g_{ij}\,,$$
$$q_{ji} = \nabla_j \kappa_i - \sigma_j \kappa_i - \kappa_j \sigma_i + \frac{1}{2}\,\sigma_a \sigma^a F_{ji}\,, \tag{2.2}$$

$$p_k^{\ h} = p_{ka} g^{ah}, \quad q_k^{\ h} = q_{ka} g^{ah}\,,$$
$$\alpha_{kj} = -(\nabla_k \kappa_j - \nabla_j \kappa_k), \quad \beta_i^{\ h} = 2(\sigma_i \kappa^h - \kappa_i \sigma^h)\,. \tag{2.3}$$

Consequently

$$q_{ji} = -\,p_{ja} F_i^{\ a}\,, \quad p_{ji} = q_{ja} F_i^{\ a}\,, \tag{2.4}$$

$$F^{kj}\alpha_{kj} = 2\,\nabla_a \sigma^a, \quad F^{kj}\beta_{kj} = -4\,\sigma_a \sigma^a\,. \tag{2.5}$$

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The vector σ_i being the gradient, we have $p_{ij} = p_{ji}$, too. We rewrite (2.1) in the form:

$$\begin{aligned} R_{kjih} &= K_{kjih} - g_{kh}p_{ji} + g_{jh}p_{ki} - p_{kh}g_{ji} + p_{jh}g_{ki} \\ &\quad - F_{kh}q_{ji} + F_{jh}q_{ki} - q_{kh}F_{ji} + q_{jh}F_{ki} \\ &\quad - \alpha_{kj}F_{ih} - \beta_{ih}F_{kj} \,. \end{aligned} \tag{2.6}$$

In (2.6) we interchange the indices k and i as well as j and h and substract from (2.6). Then we have

$$\begin{aligned} R_{kjih} - R_{ihkj} &= -F_{kh}(q_{ji}+q_{ij}) + F_{jh}(q_{ki}+q_{ik}) - F_{ji}(q_{kh}-q_{hk}) \\ &\quad + F_{ki}(q_{jh}+q_{hj}) - F_{ih}(\alpha_{kj}-\beta_{kj}) + F_{kj}(\alpha_{ih}-\beta_{ih}) \,. \end{aligned} \tag{2.7}$$

Transvecting with F^{kh}, we obtain

$$q_{ji}+q_{ij} = \frac{F^{rs}(R_{rjis} - R_{isrj})}{n+2} \tag{2.8}$$

Substituting (2.8) into (2.7), we get

$$-F_{ih}(\alpha_{kj}-\beta_{kj}) + F_{kj}(\alpha_{ih}-\beta_{ih}) = M_{kjih} \,, \tag{2.9}$$

where

$$\begin{aligned} M_{kjih} &= R_{kjih} - R_{ihkj} \\ &\quad + \frac{F^{rs}}{n+2}\left[(R_{rjis}-R_{isrj})F_{kh} - (R_{rkis}-R_{isrk})F_{jh}\right. \\ &\quad \left. + (R_{rkhs}-R_{hsrk})F_{ji} - (R_{rjhs}-R_{hsrj})F_{ki}\right] . \end{aligned} \tag{2.10}$$

Transvecting (2.9) with F^{kj} and using (2.5), we find

$$\alpha_{ih} - \beta_{ih} = -\frac{2}{n}(\nabla_a \sigma^a + 2\sigma_a\sigma^a)F_{ih} - F^{kj}M_{kjih} \,. \tag{2.11}$$

On the other hand, we find from (2.2):

$$q_{ji} - q_{ji} = \nabla_j \kappa_i - \nabla_i \kappa_j + \sigma_a \sigma^a F_{ji} \,.$$

But (2.8) gives

$$-q_{ij} = q_{ji} - \frac{F^{rs}(R_{rjis} - R_{isrj})}{n-2} \,.$$

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Therefore we have

$$\alpha_{ji} = -(\nabla_j \kappa_i - \nabla_i \kappa_j) = -2q_{ji} + \sigma_a \sigma^a F_{ji} + \frac{F^{rs}(R_{rjis} - R_{isrj})}{n-2} .$$

If we put

$$A_{ji} = \frac{F^{rs}(R_{rjis} - R_{isrj})}{n-2} , \tag{2.12}$$

we rewrite the preceding relation in the form:

$$\alpha_{ji} = -2q_{ji} + \sigma_a \sigma^a F_{ji} + A_{ji} . \tag{2.13}$$

Using (2.13), we obtain from (2.11):

$$\beta_{ji} = -2q_{ji} + \frac{2}{n}(p_a{}^a + 2\sigma_a \sigma^a) F_{ji} + B_{ji} , \tag{2.14}$$

where

$$B_{ji} = \frac{F^{rs}(R_{rjis} - R_{isrj})}{n-2} + F^{kh} M_{khji} . \tag{2.15}$$

Now, in (2.1) we contract with respect to h and k and use the skew symmetry of the tensor α_{ij}. Then putting $R_{aji}{}^a = R_{ji}$, we obtain

$$\begin{aligned} R_{ji} = {} & K_{ji} + (2-n)p_{ji} - p_a^a g_{ji} + \\ & + F_j^a q_{ai} - q_a^a F_{ji} + q_j{}^a F_{ai} + \alpha_{ja} F_i^a + \beta_{ia} F_j^a . \end{aligned} \tag{2.16}$$

On the other hand, taking into account (2.8), (2.12) and (2.2), we find

$$q_{ai} F_j^a = -p_{ji} + A_{ai} F_j^a .$$

Substituting this into (2.16), we have

$$R_{ji} - A_{ai} F_j^a = K_{ji} - np_{ji} - p_a^a g_{ji} - q_a^a F_{ji} + \alpha_{ja} F_i^a + \beta_{ia} F_j^a . \tag{2.17}$$

Transvecting (2.17) with F^{ji}, we get

$$q_a^a = \frac{1}{n}(R_{ji} F^{ji} + A_{ai} g^{ai}) .$$

Substituting this into (2.17), we obtain

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$$R_{ji} - A_{ai}F^a_j + \frac{1}{n}(R_{ab}F^{ab} + A_{ab}g^{ab})F_{ji} =$$

(2.18)

$$= K_{ji} - np_{ji} - p_a{}^a g_{ji} + \alpha_{ja}F^a_i + \beta_{ia}F^a_j \ .$$

But, taking into account (2.13) and (2.14), we find, respectively,

$$\alpha_{ja}F^a_i = -2p_{ji} + \sigma_a\sigma^a g_{ji} + A_{ja}F^a_i \ ,$$

$$\beta_{ia}F^a_j = -2p_{ji} + \frac{2}{n}(p_a{}^a + 2\sigma_a\sigma^a)g_{ji} + B_{ia}F^a_j \ .$$

Therefore, (2.18) can be written in the form:

$$\bar{R}_{ji} = K_{ji} - (n+4)p_{ji} + \left(\frac{2-n}{n}\,p_a{}^a + \frac{n+4}{n}\,\sigma_a\sigma^a\right)g_{ij} \ , \tag{2.19}$$

where we have put

$$\bar{R}_{ji} = R_{ji} + \frac{1}{n}(R_{ab}F^{ab} + A_{ab}g^{ab})F_{ji} - (A_{ai}F^a_j + A_{ja}F^a_i) - B_{ia}F^a_j \ . \tag{2.20}$$

Now, we shall use condition (1.7). Then we have

$$p_a{}^a = \nabla_a\sigma^a + \frac{n}{2}\,\sigma_a\sigma^a = \frac{n-2a}{2}\,\sigma_a\sigma^a_h \ ,$$

and (2.19) reduce to

$$\bar{R}_{ji} = K_{ji} - (n+4)p_{ji} + \frac{1}{2n}\left[(2-n)(n-2a) + 2(n+4)\right]\sigma_a\sigma^a g_{ji} \ , \tag{2.21}$$

from which, transvecting with g^{ij}, we get

$$e^{2\sigma}\bar{R} = K + \left[(n+4) - (n+1)(n-2a)\right]\sigma_a\sigma^a, \quad (\bar{R} = \bar{R}_{ji}\bar{g}^{ji})$$

i.e.

$$\sigma_a\sigma^a = \frac{e^{2\sigma}\bar{R} - K}{n+4-(n+1)(n-2a)}$$

Substituting this into (2.21) and putting

$$b = (2-n)(n-2a) + 2(n+4), \quad c = n+4-(n+1)(n-2a), \tag{2.22}$$

we find

$$p_{ji} = -\frac{\bar{R}_{ji}}{n+4} + \frac{K_{ji}}{n+4} + \frac{b}{2n(n+4)c}\,(e^{2\sigma}\bar{R} - K)g_{ji} \ . \tag{2.23}$$

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Then, using (2.4) and the fact that, in the Kähler space,

$$-K_{ja}F^a_i = K_{ai}F^a_j = K^*_{ji} ,$$

we obtain

$$(2.24)\qquad q_{ji} = -p_{ja}F^a_i = \frac{\bar{R}_{ja}F^a_i}{n+4} + \frac{K^*_{ji}}{n+4} + \frac{b}{2n(n+4)c}(e^{2\sigma}\bar{R} - K)F_{ji} .$$

Also, taking into account (2.13), we find

$$(2.25)\qquad \alpha_{ji} = -2\frac{\bar{R}_{ja}F^a_i}{n+4} - 2\frac{K^*_{ji}}{n+4} + \frac{1}{nc}(n - \frac{b}{n+4})(e^{2\sigma}\bar{R} - K)F_{ji} + A_{ji} .$$

Finally, taking into account (2.14), we have

$$(2.26)\qquad \beta_{ji} = -2\frac{\bar{R}_{ja}F^a_i}{n+4} - 2\frac{K^*_{ji}}{n+4} + \frac{1}{nc}(n-2a+4 - \frac{b}{n+4})(e^{2\sigma}\bar{R} - K)F_{ji} + B_{ji} .$$

Substituting (2.23), (2.24), (2.25) and (2.26) into (2,1), we obtain

$$\begin{aligned}
&R_{kji}{}^h + \tfrac{1}{n+4}(\delta_j{}^h\bar{R}_{ki} - \delta_k^h\bar{R}_{ji} - \bar{g}_{ji}\bar{R}_k{}^h + \bar{g}_{ki}\bar{R}_j{}^h - \\
&- F_j{}^h\bar{R}_{ka}F^a_i + F_k^h\bar{R}_{ja}F^a_i + \bar{F}_{ji}\bar{R}_{ka}\bar{F}^{ah} - \bar{F}_{ki}\bar{R}_{ja}\bar{F}^{ah} - \\
&- 2\bar{R}_{ka}F^a_jF^h_i - 2\bar{R}_{ia}\bar{F}^{ah}\bar{F}_{kj}) + \frac{\bar{R}}{n(n+4)c}\,[-b(\delta_j{}^h\bar{g}_{ki} - \\
&- \delta_k^h\bar{g}_{ji} + F_j{}^h\bar{F}_{ki} - F_k{}^h\bar{F}_{ji}) + 2n(2n+2-3a)\bar{F}_{kj}F_i^h] + \\
&+ A_{kj}F_i{}^h + B_i{}^h\bar{F}_{kj} = K_{kji}{}^h + \tfrac{1}{n+4}(\delta_j{}^hK_{ki} - \\
&- \delta_k^hK_{ji} - g_{ji}K_k{}^h + g_{ki}K_j{}^h + F_j{}^hK^*_{ki} - F_k{}^hK^*_{ji} - F_{ji}K^{*h}_k + \\
&+ F_{ki}K^{*h}_j + 2K^*_{kj}F_i{}^h + 2K^{*h}_iF_{kj}) + \frac{K}{n(n+4)c}\,[-b(\delta_j{}^hg_{ki} - \\
&- \delta_k^hg_{ji} + F_j^hF_{ki} - F_k{}^hF_{ji}) + \\
&+ 2n(2n+2-3a)F_{kj}F_i{}^h]
\end{aligned} \qquad (2.27)$$

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The tensor on the right hand side of (2.27), being independent of the function σ, isso the tensor of the left hand side. Therefore we have.

THEOREM. *If in a real n-dimensional Kähler space* ($n \geq 4$) *there exists a scalar function* σ *such that the condition (1.7) is satisfied, the thensor on the left hand side of (2.27) is invariant with respect to the complex conformal transformation (1.1).*

Let us put

$$
\begin{aligned}
H_{kji}{}^{h} &= K_{kji}{}^{h} + \frac{1}{n+4}(\delta_j^h K_{ki} - \delta_k^h K_{ji} - g_{ji}K_k^h + g_{ki}K_j{}^h + \\
&+ F_j{}^h K^*_{ki} - F_k^h K^*_{ji} - F_{ji}K_k^{*h} + F_{ki}K_j^{*h} + 2K^*_{kj}F_i{}^h + 2K_i^{*h}F_{kj}) \\
&+ \frac{K}{n(n+4)c}\,[-b(\delta_j^h g_{ki} - \delta_k^h g_{ji} + F_j{}^h F_{ki} - F_k{}^h F_{ji}) + \\
&+ 2n(2n+2-3a)F_{kj}F_i{}^h]\,.
\end{aligned}
\tag{2.28}
$$

This tensor generalizes the Bochner curvature tensor. In fact, if $a = n+2$, then $b = n(n+4)$, $c = (n+2)(n+4)$, $2n(2n+2-3a) = -2n(n+4)$ and (2.28) reduces to the Bochner curvature tensor (1.4).

Moreover, we mention the case $a = \frac{n}{2}$. Then $b = 2(n+4)$, $c = n+4$, $2n(2n+2-3a) = n(n+4)$ and $H_{kji}{}^{h}$ has the form

$$
\begin{aligned}
H_{kji}{}^{h} &= K_{kji}{}^{h} + \frac{1}{n+4}\,(\delta_j{}^h K_{ki} - \delta_k^h K_{ji} - g_{ji}K_k^h + g_{ki}K_j{}^h + \\
&+ F_j{}^h K^*_{ki} - F_k^h K^*_{ji} - F_{ji}K_k^{*h} + F_{ki}K^{*h}_j + 2K^*_{kj}F_i{}^h + 2K_i^{*h}F_{kj}) + \\
&+ \frac{K}{n+4}\,[-2(\delta_j^h g_{ki} - \delta_k^h g_{ji} + F_j^h F_{ki} - F_k^h F_{ji}) + nF_{kj}F_i^h]\,.
\end{aligned}
\tag{2.29}
$$

In section 4 we shall find another method to obtain the tensor (2.29).

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3. SOME PROPERTIES OF THE TENSOR $H_{kji}{}^{h}$

It is easy to see that the tensor $H_{kji}{}^{h}$ and $H_{kjih} = H_{kji}{}^{a} g_{ah}$ satisfy the following conditions:

$$H_{kjih} = -H_{jkih}, \quad H_{kjih} = -H_{kjhi}, \quad H_{kjih} = H_{ihkj},$$

$$H_{aji}{}^{a} = 0, \qquad H_{kja}{}^{h} F_{i}^{a} - H_{kji}{}^{a} F_{a}^{h} = 0,$$

$$H_{kjih} + H_{jikh} + H_{ikjh} = \frac{2K}{nc} (n+2-a)(F_{kj}F_{ih} + F_{ji}F_{kh} + F_{ik}F_{jh}) \tag{3.1}$$

(i.e. the condition

$$H_{kjih} + H_{jikh} H_{ikjh} = 0,$$

is satisfied only in the case when $H_{kji}{}^{h}$ is the Bochner curvature tensor or when the Kähler space is a space with zero scalar curvature).

Now, we prove two theorems concerning the tensor $H_{kji}{}^{h}$ when it differs from the Bochner curvature tensor, i.e. when $a \neq n+2$ and $a \neq \frac{n^2-4}{2(n+1)}$.

THEOREM. *The scalar curvature of the Kähler space with the parallel tensor* $H_{kji}{}^{h}$ *is constant.*

P r o o f. We find from (2.27) that

$$\begin{aligned} \nabla_t H_{kji}{}^{h} &= \nabla_t K_{kji}{}^{h} + \frac{1}{n+4} (\delta_j^h \nabla_t K_{ki} - \delta_k^h \nabla_t K_{ji} - g_{ji} \nabla_t K_k^h + g_{hi} \nabla_t K_j{}^{h} \\ &+ F_j^h \nabla_t K^*_{ki} - F_k^h \nabla_t K^*_{ji} - F_{ji} \nabla_t K^*_k{}^{h} + F_{ki} \nabla_t K^{*h}_j + \\ &+ 2F_i{}^{h} \nabla_t K^*_{kj} + 2F_{kj} \nabla_t K^{*h}_i) + \frac{\nabla_t K}{n(n+4)c} [-b(\delta_j^h g_{ki} - \\ &- \delta_k^h g_{ji} + F_j^h F_{ki} - F_k^h F_{ji}) + 2n(2n+2-3a) F_{kj} F_i{}^{h}] . \end{aligned} \tag{3.2}$$

Contracting with respect to t and h and taking into account that

$$\nabla_t K_{kji}{}^{t} = \nabla_k K_{ji} - \nabla_j K_{ki},$$

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$$F_i^a \nabla_a K^*_{kj} = \nabla_j K_{ki} - \nabla_k K_{ji} ,$$

$$\nabla_a K^a_{\ h} = \frac{1}{2} \nabla_h K, \quad \nabla_a K^{*a}_{\ k} = \frac{1}{2} F^a_k \nabla_a K ,$$

we have

$$\nabla_t H_{kji}{}^t = \frac{n}{n+4} (\nabla_k K_{ji} - \nabla_j K_{ki}) + \frac{2b-nc}{2nc(n+4)} (g_{ji}\nabla_k K -$$

$$- g_{ki}\nabla_j K + F_{ji}F^a_k \nabla_a K - F_{ki}F^a_j \nabla_a K) + \frac{4n+4-6a+c}{(n+4)c} F_{kj}F^a_i \nabla_a K .$$

Now we assume that

$$\nabla_t H_{kji}{}^h = 0 \tag{3.3}$$

Then $\nabla_t H_{kji}{}^t = 0$ too, and the above equation reduces to

$$\nabla_k K_{ji} - \nabla_j K_{ki} =$$

$$= \frac{nc-2b}{2n^2c} (g_{ji}\nabla_k K - g_{ki}\nabla_j K + F_{ji}F^a_k \nabla_a K - F_{ki}F^a_j \nabla_a K)$$

$$- \frac{4n+4-6a+c}{nc} F_{kj}F^a_i \nabla_a K .$$

Transvecting the above equation with $F^i_r F^j_s$ and taking into account that

$$(\nabla_k K_{ji} - \nabla_j K_{ki}) F^i_r F^j_s = \nabla_r K_{sk} ,$$

we have

$$\nabla_r K_{sk} = \frac{nc-2b}{2n^2c} (g_{rs}\nabla_k K - F_{rk}F^j_s \nabla_j K + F_{sr}F^a_k \nabla_a K + g_{kr}\nabla_s K) +$$

$$+ \frac{4n+4-6a+c}{nc} g_{ks}\nabla_r K . \tag{3.4}$$

At last, transvecting (3.4) with g^{sk} and taking into account (2.22), we obtain

$$[n(n^2+4n-4) - a(n^2+2n-8) - 16] \nabla_r K = 0.$$

$$[n(n^2+4n-4) - a(n^2+2n-8) - 16] \neq 0$$

because of the assumption $a \neq n+2$. Therefore

$$\nabla_r K = 0. \tag{3.5}$$

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THEOREM. *The necessary and sufficient condition that a Kähler space be the space with a parallel tensor* $H_{kji}^{\ \ h}$ *is that it be a locally symmetric space.*

P r o o f. From (3.4) and (3.5) we find

$$\nabla_i K_{jk} = 0.$$

Consequently, using (3.2) and (3.3), we get $\nabla_t K_{kji}^{\ \ h} = 0$. Conversely, if the Kähler space is symmetric, the tensor $H_{kji}^{\ \ h}$ satisfies (3.3)

4. COMPLEX CONHARMONIC CONNECTION

In the paper |3|, Y. Ishii defined the conharmonic transformation as such a comformal transformation $\bar{g}_{ij} = e^{2\sigma} g_{ij}$ which changes a harmonic function A defined by

$$g^{ij}\nabla_i\nabla_j A = 0 \tag{4.1}$$

into a function

$$\bar{A} = e^{2k\sigma} A \tag{4.2}$$

satisfying

$$\bar{g}^{ij}\bar{\nabla}_i\bar{\nabla}_j\bar{A} = 0,$$

where k is a suitable constant and $\bar{\nabla}_i$ is the operator of the covariant differentiation with respect to

$$\overline{\left\{ \begin{matrix} h \\ ji \end{matrix} \right\}} = \left\{ \begin{matrix} h \\ ji \end{matrix} \right\} + \delta_j^{\ h}\sigma_i + \delta_i^h\sigma_j - g_{ij}\sigma^h .$$

The main purpose of the present section is to find, using the connection (1.2), the complex analogue of Ishii's results.

We assume that condition (4.1) is satisfied and we seek the condition upon σ in order that the function (4.2) satisfies

$$\bar{g}^{ij}D_jD_i\bar{A} = 0. \tag{4.3}$$

First, we have

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$$D_i\bar{A} \equiv \bar{A}_i = \partial_i \bar{A} = e^{2k\sigma}(2kA\sigma_i + A_i), \quad A_i = \partial_i A.$$

Then, taking into account (1.2), we get

$$\begin{aligned} D_j\bar{A}_i = e^{2k\sigma}\big[&2kA\nabla_j\sigma_i + \nabla_j A_i + 4k(k-1)A\sigma_j\sigma_i + (2k-1)\sigma_j A_i + \\ &+ (2k-1)\sigma_i A_j + 2kAg_{ij}\sigma^a\sigma_a + g_{ij}\sigma^a A_a \\ &- 2kAF^a_{j}\sigma_a\kappa_i - 2kAF^a_{i}\sigma_a\kappa_j + 2kAF_{ji}{}^{\kappa a}\sigma_a \\ &- F^a_{j}A_a\kappa_i - F^a_{i}A_{a\,j} + F_{ji}{}^{\kappa a}A_a\big] . \end{aligned}$$

Therefore

$$\begin{aligned} \bar{g}^{ij}D_j\bar{A}_i = e^{2(k-1)\sigma}\big[&g^{ij}\nabla_j A_i + 2kAg^{ij}\nabla_j\sigma_i + 2k(2k-2+n)A\sigma_a\sigma^a \\ &+ (4k-2+n)\sigma_a A^a - 4kAg^{ij}F^a_{j}\sigma_a\kappa_i - 2g^{ij}F^a_{j}\kappa_i A_a\big] \end{aligned}$$

Consequently, using the conditions (4.1) and (4.3), we find

$$\begin{aligned} &2kAg^{ij}\nabla_j\sigma_i + 2k(2k-2+n)A\sigma_a\sigma^a + (4k-2+n)\sigma_a A^a - \\ &- 4kAg^{ij}F^a_{j}\sigma_a\kappa_i - 2g^{ij}F^a_{j}\kappa_i A_a = 0 , \end{aligned}$$

or

$$(4.4) \qquad 2kAg^{ij}\nabla_j\sigma_i + 2k(2k+n)A\sigma_a\sigma^a + (2k+n)\sigma_a A^a = 0,$$

because of

$$-g^{ij}F^a_{j}\kappa_i A_a = \sigma^a A_a ,$$

and

$$-g^{ij}F^a_{j}\sigma_a\kappa_i = \sigma_a\sigma^a .$$

If we determine the number k by $k = -\frac{n}{4}$, (4.4) reduces to

$$(4.5) \qquad 2g^{rs}\nabla_r\sigma_s + n\sigma_a\sigma^a = 0 .$$

(4.5) is the required condition upon σ.

We call an affine connection (1.2) where the conditions (1.3) and (4.5) are satisfied a complex conharmonic connection.

The tensor formed with the curvature tensor of the co-

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mplex conharmonic connection and independent of the function σ can be called the complex conharmonic curvature tensor. On the other hand, the condition (1.7) reduces to the condition (4.5), if $a = \frac{n}{2}$. It follows that (2.29) is the complex conharmonic curvature tensor. In other words, the tensor (2.28) generalizes the complex conharmonic curvature tensor, too.

5. GENERALIZATION OF THE CONTACT BOCHNER CURVATURE TENSOR

Now, we compute the curvature tensor $R_{kji}{}^{h}$ of the contact conformal connection (1.5), and find [2]:

$$
\begin{aligned}
R_{kji}{}^{h} &= K_{kji}{}^{h} \\
&- (\delta_k^h - \eta_k\eta^h)p_{ji} + (\delta_j^h - \eta_j\eta^h)p_{ki} - p_k{}^h(g_{ji}-\eta_j\eta_i) + \\
&+ p_j{}^h(g_{ki} - \eta_k\eta_i) - \phi_k{}^h q_{ji} + \phi_j{}^h q_{ki} - q_k{}^h\phi_{ji} + \\
&+ q_j{}^h\phi_{ki} - \alpha_{ki}\phi_i^h - \phi_{kj}\beta_i{}^h + \phi_k{}^h\phi_{ji} - \\
&- \phi_j{}^h\phi_{ki} - 2\phi_{kj}\phi_i{}^h ,
\end{aligned}
\tag{5.1.}
$$

where

$$
\begin{aligned}
p_{ji} &= \nabla_j\sigma_i - \sigma_i\sigma_j + (\kappa_j-\eta_j)(\kappa_i-\eta_i) + \frac{1}{2}\sigma_a\sigma^a(g_{ji} - \eta_j\eta_i) , \\
q_{ji} &= \nabla_j\kappa_i - \sigma_j(\kappa_i-\eta_i) - \sigma_i(\kappa_j-\eta_j) + \frac{1}{2}\sigma_a\sigma^a\phi_{ji} , \\
\alpha_{kj} &= -(\nabla_k\kappa_j - \nabla_j\kappa_k), \qquad \beta_i{}^h = (2\sigma_i\kappa^h - \kappa_i\sigma^h) .
\end{aligned}
$$

Using the condition (1.7) where, now, $a \neq \frac{2(m^2-1)}{2m+1}$, and proceding in a similar manner as in section 2, we find

$$
\begin{aligned}
p_{ji} = &- \frac{R_{ji}}{2(m+2)} + \frac{K_{ji}}{2(m+2)} + \left[E + (e^{2\sigma}\bar{R} - K)D\right]g_{ij} \\
&- \left[E - \frac{4}{2(m+2)} + (e^{2\sigma}\bar{R} - K)D\right]\eta_i\eta_j
\end{aligned}
\tag{5.2}
$$

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$$(5.3)\qquad q_{ji} = \frac{\bar{R}_{ja}\phi_i^a}{2(m+2)} - \frac{K_{ja}\phi_i^a}{2(m+2)} + \left[E + (e^{2\sigma}\bar{R} - K)D\right]\phi_{ji}$$

$$\alpha_{ji} = -\frac{\bar{R}_{ja}\phi_i^a}{m+2} + \frac{K_{ja}\phi_i^a}{m+2}$$

$$(5.4)\qquad - \left[(2E + \frac{2m}{C}) + e^{2\sigma}(2D - \frac{1}{C})\bar{R} - (2D - \frac{1}{C})K\right]\phi_{ji} + A_{ji},$$

$$\beta_{ji} = -\frac{\bar{R}_{ja}\phi_i^a}{m+2} + \frac{K_{ja}\phi_i^a}{m+2} + B_{ji}$$

$$(5.5)\qquad - \left[(2E + 2\frac{m+2-a}{C}) + e^{2\sigma}\bar{R}(2D - \frac{m+2-a}{mC}) - K(2D - \frac{m+2-a}{mC})\right]\phi_{ji},$$

where $2m+1$ is the dimension of the considered Sasakian manifold, and

$$A_{ji} = \frac{(R_{rjis} - R_{isrj})\phi^{rs}}{2-2m},$$

$$B_{ji} = A_{ji} - \frac{M_{rsji}\phi^{rs}}{2m},$$

$$M_{rjih} = R_{kjih} - R_{ihkj} +$$

$$+ \phi_{kh}\frac{(R_{rjis} - R_{isrj})\phi^{rs}}{2-2m} - \phi_{jh}\frac{(R_{rkis} - R_{isrk})\phi^{rs}}{2-2m}$$

$$+ \phi_{ji}\frac{(R_{rkhs} - R_{hsrk})\phi^{rs}}{2-2m} - \phi_{ki}\frac{(R_{rjhs} - R_{hsrj})\phi^{rs}}{2-2m}$$

$$\bar{R}_{ji} = R_{ji} - A_{ja}\phi_i^a - A_{ai}\phi_j^a - B_{ib}\phi_j^b - \frac{(R_{rs} - A_{as}\phi_r^a)\phi^{rs}}{}\phi_{ji}$$

$$R_{ji} = R_{aji}{}^a, \quad \bar{R} = \bar{R}_{ji}g^{ji}$$

$$C = 4(1-m^2) + 2(2m+1)a$$

$$E = \frac{C - [2+2m-m^2+a(m-1)]}{(m+2)C}, \qquad D = \frac{2+2m-m^2+a(m-1)}{2m(m+2)C}.$$

Substituting (5.2), (5.3), (5.4) and (5.5) into (5.1), we obtain that the tensor

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$$H_{kji}{}^{h} = R_{kji}{}^{h}$$

$$- (\delta_k^h - \bar{\eta}_k\bar{\eta}^h)\Big[\frac{\bar{R}_{ji}}{2(m+2)} - \bar{R}(\bar{g}_{ij} - \bar{\eta}_i\bar{\eta}_j)D\Big] + (\delta_j^h - \bar{\eta}_j\bar{\eta}^h)\Big[\frac{\bar{R}_{ki}}{2(m+2)} -$$

$$- \bar{R}(\bar{g}_{ki} - \bar{\eta}_k\bar{\eta}_i)D\Big] - (\bar{g}_{ji} - \bar{\eta}_j\bar{\eta}_i)\Big[\frac{\bar{R}_k{}^h}{2(m+2)} - \bar{R}(\delta_k^h - \bar{\eta}_k\bar{\eta}^h)D\Big] +$$

$$+ (\bar{g}_{ki} - \bar{\eta}_k\bar{\eta}_i)\Big[\frac{\bar{R}_j{}^h}{2(m+2)} - \bar{R}(\delta_j^h - \eta_j\eta^h)D\Big] + \phi_k{}^h\Big(\frac{\bar{R}_{ja}\phi_i^a}{2(m+2)} + \bar{R}\bar{\phi}_{ji}D\Big) -$$

$$- \phi_j{}^h\Big(\frac{\bar{R}_{ka}\phi_i^a}{2(m+2)} + \bar{R}\bar{\phi}_{ki}D\Big) + \bar{\phi}_{ji}\Big(\frac{\bar{R}_{ka}\bar{\phi}^{ha}}{2(m+2)} + \bar{R}\phi_j{}^hD\Big) - \bar{\phi}_{ki}\Big(\frac{\bar{R}_{ja}\bar{\phi}^{ha}}{2(m+2)}$$

$$+ \bar{R}\phi_j{}^hD\Big) - \phi_j{}^h\Big[2\frac{\bar{R}_{ka}\phi_j^a}{2(m+2)} + \bar{R}\Big(2D - \frac{1}{C}\Big)\bar{\phi}_{kj} - A_{kj}\Big]$$

$$- \bar{\phi}_{kj}\Big[2\frac{\bar{R}_{ia}\bar{\phi}^{ha}}{2(m+2)} + \bar{R}\Big(2D - \frac{m+2-a}{mC}\Big)\phi_i{}^h - B_i{}^h\Big]$$

is independent of the function σ satisfying (1.7). In fact

$$H_{kji}{}^{h} = K_{kji}{}^{h} - (\delta_k^h - \eta_k\eta^h)\Big[\frac{K_{ji}}{2(m+2)} + (E-KD)g_{ji} - \Big(E - \frac{2}{m+2} - KD\Big)\eta_i\eta_j\Big]$$

$$+ (\delta_j{}^h - \eta_j\eta^h)\Big[\frac{K_{ki}}{2(m+2)} + (E-KD)g_{ki} - \Big(E - \frac{2}{m+2} - KD\Big)\eta_k\eta_i\Big]$$

$$- (g_{ji} - \eta_j\eta_i)\Big[\frac{K_k{}^h}{2(m+2)} + (E-KD)\delta_k^h - \Big(E - \frac{2}{m+2} - KD\Big)\eta_k\eta^h\Big] \qquad (5.6)$$

$$+ (g_{ki} - \eta_k\eta_i)\Big[\frac{K_j{}^h}{2(m+2)} + (E-KD)\delta_j^h - \Big(E - \frac{2}{m+2} - KD\Big)\eta_j\eta^h\Big]$$

$$- \phi_k^h\Big[-\frac{K_{ja}\phi_i^a}{2(m+2)} + (E-KD)\phi_{ji}\Big] + \phi_j{}^h\Big[-\frac{K_{ka}\phi_i^a}{2(m+2)} + (E-KD)\phi_{ki}\Big]$$

$$- \phi_{ji}\Big[-\frac{K_{ka}\phi^{ha}}{2(m+2)} + (E-KD)\phi_k{}^h\Big] + \phi_{ki}\Big[-\frac{K_{ja}\phi^{ha}}{2(m+2)} + (E-KD)\phi_j{}^h\Big]$$

$$- \frac{1}{m+2}(K_{ia}\phi^{ha}\phi_{kj} + K_{ka}\phi_j^a\phi_i{}^h) + \Big[4E + \frac{4m+4-2a}{C} -$$

$$- \Big(4D - \frac{2m+2-a}{mC}\Big)K\Big]\phi_{kj}\phi_i{}^h + \phi_k{}^h\phi_{ji} - \phi_j{}^h\phi_{ki} - 2\phi_{kj}\phi_i{}^h \quad .$$

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If $a = 2(m+1)$, then

$$C = 4(m+1)(m+2), \qquad D = \frac{1}{8(m+1)(m+2)},$$

$$E = \frac{3m+4}{4(m+1)(m+2)}, \qquad E - \frac{2}{m+2} = -\frac{5m+4}{4(m+1)(m+2)}$$

and the tensor $H_{kji}{}^{h}$ has the form (1.6), i.e. $H_{kji}{}^{h}$ reduces to the contact Bochner curvature tensor. Therefore, the tensor $H_{kji}{}^{h}$ generalizes the contact Bochner curvature tensor. It generalizes the contact conharmonic curvature tensor, too.

In fact, we assume that condition (4.1) is satisfied and we seek the condition upon σ in order that function (4.2) satisfies

$$\bar{g}^{ij}\mathcal{D}_j\mathcal{D}_i\bar{A} = 0,$$

where $\mathcal{D}_i$ is the operator of the covariant differentiation with respect to the contact conformal connection (1.5). Proceeding in a similar manner as in section 4, we find that the required condition is:

$$(5.7) \qquad \nabla_a\sigma^a + \frac{m^2+m+1}{m}\sigma_a\sigma^a = 0$$

We shall an affine connection (1.5), where the condition (5.7) is satisfied, a contact conharmonic connection. The tensor formed with the curvature tensor of the contact conharmonic connection and independent of the function σ can be called the contact conharmonic curvature tensor. On the other hand, the condition (5.7) is the special case of the condition (1.7). Therefore, the tensor (5.6) where $a = \frac{m^2+m+1}{m}$ is the contact conharmonic curvature tensor.

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REZIME

JEDNO UOPŠTENJE BOHNEROVOG I KONTAKTNO-BOHNEROVOG TENZORA KRIVINE

U § 2 posmatra se Kähler-ov prostor koji dopušta takvu skalarnu funkciju σ da je zadovoljen uslov (1.7). Koristeći tenzor krivine kompleksne konformne koneksije (1.2), konstruisan je tenzor $H_{kji}{}^{h}$ tj. tenzor (2.28), koji ne zavisi od funkcije σ. U slučaju kad je $a=n+2$, tenzor $H_{kji}{}^{h}$ se svodi na Bohnerov tenzor. U § 3 dokazane su neke teoreme o tenzoru $H_{kji}{}^{h}$. U § 4 odredjen je tenzor kompleksno konharmonijske krivine koji je takodje jedan specijalan slučaj tenzora $H_{kji}{}^{h}$.

U § 5 posmatra se $(2m+1)$-dimenzioni prostor Sasaki-ja koji dopušta takvu skalarnu funkciju σ da je zadovoljen uslov (1.7) uz ograničenje $a \neq \frac{2(m^2-1)}{2m+1}$. Koristeći tenzor krivine kontaktne konformne transformacije (1.5), konstruisan je tenzor $H_{kji}{}^{h}$ tj. tenzor (5.6).Taj tenzor je uopštenje i kontaktno-Bochner-ovog tenzora krivine i tenzora kontaktno-konharmonijske krivine.

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ON A STRUCTURE DEFINED BY A TENSOR FIELD f OF THE TYPE (1,1) SATISFYING $f^{2\cdot 2^q+1}-f=0$

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ABSTRACT

In this paper was chose an adapted frame for f(2k+1,-1)-structure and a matrix of tensors g_{ij} and f_i^j with respect to this adapted frame. Given is the necessary and sufficient condition for an n-dimensional manifold M^n to admit a tensor field f of the type (1,1) and the rank r such that $f^{2\cdot 2^q+1} - f = 0$, $f^{2i+1}-f \neq 0$ for $1 \leq i < 2^q$, $q \in N$.

1. Let us first observe the structure f with satisfies the condition $f^{2k+1} - f = 0$.

DEFINITION 1. *Let M^n be a differentiable manifold of the class C^∞ and let there be given a tensor field $f \neq 0$ of the type (1,1) and of the class C^∞ such that*

(1.1) $$f^{2k+1} - f = 0, \quad f^{2i+1} - f \neq 0 \quad \text{for} \quad 1 \leq i < k$$

where k is a fixed positive integer greater than 1. Let rank f = r be constant. We call such a structure an (2k+1,-1)-structure or an f-structure of rank r and of 2k+1 -degree.

THEOREM 1.1. *For a tensor field f, $f \neq 0$ satisfying (1.1), the operators*

(1.2) $$m = 1 - f^{2k} \quad \text{and} \quad l = f^{2k}$$

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(1.1) *denoting the identity operator applied to the tangent space at a point of the manifold, are complementary projection operators.*

P r o o f. We have

$$\ell + m = 1, \quad \ell^2 = \ell, \quad m^2 = m, \quad m\ell = \ell m = 0$$

by virtue of (1.1) which proves the Theorem.

Let L and M be the complementary distributions corresponding to the operators ℓ and m, respectively. If the rank $f = r$ is constaht, then $\dim L = r$ and $\dim M = n - r$.

THEOREM 1.2. *For* f *satisfying* (1.1) *and* ℓ, m, *defined by* (1.2) *we have*

$$\ell f = f\ell = f, \quad mf = fm = 0, \quad f^2 m = 0.$$

P r o o f. Trivial.

THEOREM 1.3. *For* f *satisfying* (1.1) *and* m, *defined by* (1.2) *we have*

$$(1.3) \qquad (m+f^k)^2 = 1, \quad fm = mf = 0 .$$

P r o o f. Trivial.

THEOREM 1.4. *Suppose that there is given on* M^n *a projection operator* m *and that there exists a tensor field* f *such that* (1.3) *is satisfied, then* f *satisfies* (1.1).

PROPOSITION 1.1. *Let an* f*-structure of rank* r *and of the* 2k+1 *-degree be given on* M^n, *then* $f^{2k}\ell = \ell$, *and* $f^{2k} m = 0$. *Then* f^k *acts on* L *as almost a product structure operator and on* M *as almost a tangent structure operator.*

2. We shall now introduce a local coordinate system in the manifold and denote by f_i^p, ℓ_i^p, m_i^p the local components of the tensor f, ℓ, m, respectively. We shall also introduce a positive definite Riemannian metric g in the manifold and take r as mutually orthogonal unit vector u_a^p (a,b,c = 1,2,...,r) in L and n-r as mutually orthogonal unit vector u_A^p (A,B,C = r+1,...,n)

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in M. We have

$$\ell^p_i u^i_b = u^p_b,\ \ell^p_i u^i_B = 0,\ m^p_i u^i_b = 0,\ m^p_i u^i_B = u^p_B .$$

We also have $f^p_i u^i_B = 0$. If we denote by (v^a_i, v^A_i) the matrix inverse to (u^p_b, u^p_B), then v^a_i and v^A_i are both components of linearly independent covariant vectors and satisfy

$$(2.1)\quad v^a_i u^i_b = \delta^a_b,\ v^a_i u^i_B = 0,\ v^A_i u^i_B = 0$$

$$v^A_i u^i_B = \delta^A_B,\ v^a_u u^p_a + v^A_i u^p_A = \delta^p_i .$$

We can easily prove that

$$(2.2)\quad \ell^p_i v^a_p = v^a_i,\ \ell^p_i v^a_p = 0,\ m^p_i v^\ell_p = 0$$

$$m^p_i v^A_p = v^A_i .$$

From $mf = 0$, we find $f^p_i v^A_p = 0$. From $\ell^p_i u^i_a = u^p_a$, we find $\ell^p_i = v^a_i u^p_a$. From (2.1) and (2.2), we also find $m^p_i = v^A_i u^p_a$.

If we put $a_{ij} = v^a_j v^a_i + v^A_j v^A_i$, then a_{ij} is a globally positive definite Riemannian metric with respect to which (u^p_b, u^p_B) forms an orthogonal frame such that $v^a_j = a_{ij} u^i_A$ and $v^A_j = a_{ij} u^i_A$.

If we put $\ell_{ij} = \ell^t_j a_{ti}$, $m_{ij} = m^t_j a_{ti}$, we find $\ell_{ji} = v^a_j v^a_i$, $m_{ji} = v^A_j v^A_i$. Consequently $\ell_{ji} + m_{ji} = a_{ji}$.

We can easily verify the following relations:

$$\ell^t_j \ell^s_i a_{ts} = \ell_{ji},\ \ell^t_j m^s_i a_{ts} = 0,\ m^t_j m^s_i a_{tj} = m_{ji} .$$

For any two vectors x,y with components x^i, y^i, let us put

$$m^*(x,y) = m_{st} x^s y^t,\quad a(x,y) = a_{st} x^s y^t ,$$

$$g(x,y) = \frac{1}{k}\,(a(x,y) + (\sum_{z=1}^{2k-1} (f^z x, f^z y))a + m^*(x,y)) .$$

Then we have

$$m^*(u_A, u_a) = a(u_A, u_a) = a(fu_A, fu_a) = a(f^2 u_A, f^2 u_a) =$$

$$= \ldots = a(f^{2k-1} u_A, f^{2k-1} u_a) = 0,\quad g(u_A, u_a) = 0.$$

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Thus L and M are orthogonal with respect to g. We also have

$$m^*(u_a,u_b)=0, \quad a(f^{2k}u_a,f^{2k}u_b)=a(u_a,u_b) ,$$

Hence

$$g(u_a,u_b)=\frac{1}{k}\,(a(u_a,u_b)+(\sum_{z=1}^{2k-1}(f^z u_a,f^z u_b))a) ,$$

$$g(fu_a,fu_b)=\frac{1}{k}\,(a(\sum_{z=1}^{2k}(f^z u_a,f^z u_b)))=$$

$$=\frac{1}{k}\,(a(\sum_{z=1}^{2k-1}(f^z u_a,f^z u_b))+a(u_a,u_b),$$

That is

$$g(x,y) = g(fx, fy)$$

for all vectors x,y in L.

We assume that $f_L^i = f^i\ell/L$ $(i < 2k)$ is not the identity operator of L. Then f_L is a linear transformation of L with minimal polynominal $x^{2k}-1=0$. (We know that $f^{2k}=1$ on L). The polynominal $(x^k-1)(x^k+1)=0$ has simple roots

$$e^{\frac{2\pi i}{k}}, e^{3\frac{2\pi i}{k}}, \ldots, e^{(2k-1)\frac{2\pi i}{k}}, e^{2\frac{2\pi i}{k}}, e^{4\frac{2\pi i}{k}}, \ldots, e^{2k\frac{2\pi i}{k}}$$

The eigen vectors which correspond to these eigenvalues are e_1, $e_3,\ldots,e_{2k-1}$, e_2, $e_4,\ldots,e_{2k}$, respectively. Let us denote by L_2 the vector space generazed by vectors $e_1,e_3,\ldots,e_{2k-1}$ and by L_1 the vector space generized by vectors $e_2,e_4,\ldots,e_{2k}$.

$$f^k = -1 \quad \text{on} \quad L_2, \quad f^k = 1 \quad \text{on} \quad L_1 .$$

For $x \in L_1$ and $y \in L_2$, we have

$$g(x,y)=g(fx,fy)=g(f^k x,f^k y)=g(x,-y)=-g(x,y) .$$

Hence L_1, L_2 are orthogonal with respect to the metric g.

We assume that $f^j \neq 1$ on L_1, $j<k$ and $f^j \neq -1$ on L_2, $j<k$. Then f is a linear transformation of L_2 with the minimal polynomial $x^k+1=0$, with the eigenvalue $\sqrt[k]{-1}$, to which correspond the eigen vector e_1', e_2',...,e_k' and $L_2 = L_2^1 + L_2^2 + \ldots + L_2^k$ where L_2^s are subspaces of L_2 generized by the vectors e_s'.

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It is also an f linear transformation on L_1 with the minimal polynomial $x^k-1=0$, with the eigenvalue $\sqrt[k]{1}$, to which correspond the eigen vectors $e'_{k+1}, e'_{k+2}, \ldots, e'_{2k}$. Now, $L_1 = L_1^{k+1} + L_1^{k+2} + \ldots + L_1^{2k}$, where L_1^{k+1} are subspaces of L_1 generized by the vectors e'_{k+s}.

L_1^{k+p} and L_1^{k+r} are orthogonal with respect to g only if $k=2^q$, $q \in N$, which is then shown by induction. In the following text $k=2^q$, $q \in N$.

Let $x \in L_2$. Then $f(x) \in L_2$. We have

$$g(x,fx) = g(fx,f^2x) = \ldots = g(f^kx,f^{k+1}x) = g(-x,fx) = -g(x,fx),$$

Hence $L_2 = \bar{L}_2 \oplus f(\bar{L}_2)$ and $\bar{L}_2$ and fL_2 are orthogonal spaces with respect to g.

We assume that $\dim L_2 = d$, $\dim f\bar{L}_2 = s$, then

$\dim L_2 = d+s = 2p$, $\dim L_1 = r-d-s = r-2p$. Let $r=(q+2)2p$.

In |4| the following Theorem is proved:

THEOREM. *If*

$$f^{\bar{k}} = \begin{bmatrix} 0 & E_p \\ -E_p & 0 \end{bmatrix}$$

then $\bar{k} \leq p$ *and* p *is divisible by* $\bar{k}$, $(p = s\bar{k})$.

In our case, there is such a state in the space L_2. $(d+s=2p)$.

Let $u_1, \ldots, u_{2p}$ be an orthogonal basis of L_2 $(p=s\cdot 2^{q-1})$, and $u_{2p+1}, u_{2p+2}, \ldots, u_{r-2p}$ be an orthogonal basis of L_1, both with respect to g, then $u_1, \ldots, u_{2p}, u_{2p+1}, \ldots, u_{r-2p}$ is an orthogonal basis of L such that

for: $L_2: f(u_i) = u_{i+\frac{p}{2^q}}, f(u_{i+2p-\frac{p}{2^q}}) = -u_i$, $i=1,2,\ldots,p$

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$$\text{for } L_1:\begin{cases} f(u_{2p+i}) = u_{2p+i+\frac{p}{2^{q-1}}}, f(u_{4p+i-\frac{p}{2^{q-1}}}) = -u_{2p+i}, & i=1,2,\dots,p \\ & i=1,2,\dots,p \\ f(u_{4p+i}) = u_{4p+i+\frac{p}{2^{q-2}}}, f(u_{6p+i-\frac{p}{2^{q-2}}}) = -u_{4p+i}, & i=1,2,\dots,p \\ & i=1,2,\dots,p \\ \vdots \\ f(u_{2qp+i}) = u_{2pq+i}, f(u_{2(q+1)p-i}) = -u_{2p(q+1)-i} & \\ & i=1,2,\dots,p \end{cases}.$$

Next, we choose in M an orthogobal basis $u_{r+1},\dots,u_n$ with respect to g. Then with respect to the orthogonal frame $u_1,\dots,u_n$ the tensors g_{ij} and f_i^j have the components

(2.3)

$$f = \begin{bmatrix} 0 & E_{2p-\frac{2p}{2^q}} & & & & & & & \\ -E_{\frac{2p}{2^q}} & 0 & & & & & & & \\ & & \ddots & & & & & & \\ & & & 0 & E_{2p-\frac{p}{2}} & & & & \\ & & & -E_{\frac{p}{2}} & 0 & & & & \\ & & & & & 0 & E_p & & \\ & & & & & -E_p & 0 & & \\ & & & & & & & E_p & 0 \\ & & & & & & & 0 & -E_p \\ & & & & & & & & & 0_{n-r} \end{bmatrix} \Bigg\} q+1$$

$$g = \begin{bmatrix} E_{2p} & & & \\ & \ddots & & \\ & & E_{2p} & \\ & & & E_{n-r} \end{bmatrix} \Bigg\} q+1$$

We call such a frame an adapted frame of f(2k+1,-1) structure.

Let $\bar{u}_1,\dots,\bar{u}_n$ be another adapted frame with respect to which the metric tensor g and the tensor f have the same components as (2.3). We put $\bar{u}_i = \gamma_i^j u_j$ then we can find that γ has the form

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$$(2.4)\qquad \gamma = \begin{bmatrix} S_{(\frac{2p}{2^q})} & & & & & & \\ & S_{(\frac{2p}{2^{q-1}})} & & & & & \\ & & \ddots & & & & \\ & & & S_{(\frac{p}{2})} & & & \\ & & & & \begin{matrix} A_p & B_p \\ -B_p & A_p \end{matrix} & & \\ & & & & & {}^{0}{}_{2p} & \\ & & & & & & {}^{0}{}_{n-r} \end{bmatrix}$$

where $S_{(\frac{2p}{i})}$, $i=2,4,\ldots,2^q$ is a matrix of format 2p and has the form

$$(2.5)\qquad S_{(\frac{2p}{i})} = \begin{bmatrix} A_1 & A_2 & A_3 & A_4 & \cdots & A_i \\ -A_i & A_1 & A_2 & A_3 & \cdots & A_{i-1} \\ -A_{i-1} & -A_i & A_1 & A_2 & \cdots & A_{i-2} \\ \vdots & & & & & \\ -A_3 & -A_4 & -A_5 & & & A_2 \\ -A_2 & -A_3 & -A_4 & & \cdots & A_1 \end{bmatrix}$$

where each matrix A_t, $t=1,\ldots,i$ has a format $(\frac{2p}{i}) \times (\frac{2p}{i})$, i.e. s xs.

Let $\bar{S}_{(\frac{2p}{i})}$ be the tangent group defined by $S_{(\frac{2p}{i})}$. Then we can say that the group of the tangent bundle of the manifold can be reduced to

$$\bar{S}_{(\frac{2p}{2^q})} \times \bar{S}_{(\frac{2p}{2^{q-1}})} \times \ldots \times \bar{S}_{(\frac{2p}{4})} \times U_p \times O_{2p} \times O_{n-r}$$

THEOREM 2.1. *A necessary and sufficient condition for an n-dimensional manifold M^n to admit a tensor field $f \neq 0$ of the type (1,1) and of rank r, such $f^{2\cdot 2^{q+1}} - f = 0$, is that*

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i) $r = (q+1)2p$, ii) $p = s \cdot 2^q = s \cdot k$ *and* iii) *the group of the tangent bundle of the manifild be reduced to the group*

$$\bar{S}_{(\frac{2p}{2^q})} \times \bar{S}_{\frac{2p}{2^{q-1}}} \times \ldots \times \bar{S}_{(\frac{2p}{4})} \times U_p \times O_{2p} \times O_{n-r} .$$

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REZIME

O STRUKTURI KOJA JE DEFINISANA TENZORSKIM POLJEM f TIPA (1,1) KOJE ISPUNJAVA USLOV $f^{2\cdot 2^q+1} - f = 0$

U radu je definisana f(2k+1,-1) struktura, izabran adaptirani reper za tu strukturu, nadjene su martice za tenzore g_{ij} i f_i^j u odnosu na taj adaptirani reper. Dat je potreban i dovoljan uslov da se n-dimenzionalna mnogostrukost može snabdeti tenzorskim poljem f tipa (1,1) i ranga r da je $f^{2\cdot 2^q+1}-f=0$, $f^{2i+1}-f\neq 0$ za $1 \leq i < 2^q$, $q \in N$.

TEOREMA 2.1. *Potreban i dovoljan uslov da se n-dimenzionalna mnogostrukost* M^n *može snabdeti tenzorskim poljem* $f \neq 0$ *tipa* (1,1) *i ranga* r, *tako da je* $f^{2\cdot 2^q+1} - f = 0$ *je*

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i) $r = (q+1)2p$

ii) $p = s \cdot 2^q = sk$

iii) *grupa tangentnog bandla mnogostrukosti se reducira do*

$$\bar{S}_{(\frac{2p}{2^q})} \times \bar{S}_{(\frac{2p}{2^{q-1}})} \times \ldots \times \bar{S}_{(\frac{2p}{4})} \times U_{(p)} \times O_{(2p)} \times O_{(n-r)}$$

gde je $\bar{S}_{(\frac{2p}{i})}$ *tangentna grupa definisana sa* $S_{(\frac{2p}{i})}$ *čiji je oblik dat formulom* (2.5).

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WEYL-ŌTSUKI SPACES OF THE SECOND KIND WITH A SPECIAL TENSOR P

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ABSTRACT

In this paper, we define Weyl-Ōtsuki spaces of the second kind and order n (denoted SW-O_n) with a tensor P in the form $P^i_s = S(x)\delta^i_s$, where S(x) is a non-zero and non-linear C^r $(r \geq 3)$ function. We establish some relations between the curvature tensor and the conformal curvature-like tensor in SW-O_n and analogous objects in an adjoint Riemannian space.

1. The basic objects of spaces investigated by T.Ōtsuki (|1|) are two objects of connection, denoted by $'\Gamma^i_{jk}$ and $''\Gamma^i_{jk}$, and a regular tensor field P of the type (1,1). $'\Gamma^i_{jk}$ and $''\Gamma^i_{jk}$ are objects of two affine connections $'\Gamma$ and $''\Gamma$ named the contravariant and covariant part of the regular general connection respectively.

The basic covariant differentiation is defined by the following formulas

$$v_{i|k} = \frac{\partial v_i}{\partial u^k} - {}''\Gamma^s_{ik} v_s$$

$$v^i_{|k} = \frac{\partial v^i}{\partial u^k} + {}'\Gamma^i_{sk} v^s$$

The covariant differentiation with respect to the general regular connection is defined for an arbitrary tensor in the following way:

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$$(1.1)\qquad T^{i}_{jk,\ell} = P^{i}_{a}T^{a}_{bc|\ell}P^{b}_{j}P^{c}_{k}\ .$$

Connections $'\Gamma$ and $''\Gamma$ are mutually dependent by relation

$$(1.2)\qquad Q^{i}_{j|k} = 0$$

where Q denotes the inverse of P.

The Weyl-Ōtsuki space (W-O_n) is defined by A.Moór (|2|, |3|). This is an Ōtsuki space provided with a symmetric positive metric tensor g_{ij} and a recurrence vector γ_k. The W-O_n space satisfied the following conditions:

a) $g_{ij,k} = \gamma_k g_{ij}$ (the metric tensor is recurrent),

b) $''\Gamma^{i}_{jk} = ''\Gamma^{i}_{kj}$ (the covariant part of a general regular connection is symmetric),

c) $P_{ij} = g_{ia}P^{a}_{j} = g_{ja}P^{a}_{i} = P_{ji}$.

In W-O_n spaces, the coefficients of connection $''\Gamma$ have the form

$$''\Gamma^{i}_{jk} = \{^{i}_{jk}\} - \frac{1}{2}\,g^{is}(\gamma_j g_{ab}Q^{a}_{s}Q^{b}_{k} + \gamma_k g_{ab}Q^{a}_{s}\ Q^{b}_{j} - \gamma_s g_{ab}Q^{a}_{j}Q^{b}_{k})$$

where $\{^{i}_{jk}\}$ denote the Christoffel symbols with respect to g_{ij}.

M.Prvanović (|4|,|5|) has defined another kind of metric Ōtsuki space with a recurrent metric tensor; these Ōtsuki spaces with a regular general connection satisfy the following conditions :

a′) $g_{ij,k} = \gamma_k m_{ij}$, where m_{ij} is a symmetric tensor field,

b′) the connection $'\Gamma$ is symmetric ($'\Gamma^{i}_{jk} = '\Gamma^{i}_{kj}$),

c′) $P_{ij} = g_{ia}P^{a}_{j} = g_{ja}P^{a}_{i} = P_{ji}$.

The coefficients of connections $''\Gamma$ and $'\Gamma$ can be calculated by the following formulas:

$$(1.3)\qquad \begin{aligned} ''\Gamma^{i}_{jk} &= {}''\overset{m}{\Gamma}{}^{i}_{jk} + \frac{1}{2}\,g^{is}(\gamma_q m_{pk}Q^{q}_{s}Q^{p}_{j} - \gamma_k m_{pq}Q^{p}_{s}Q^{q}_{j} - \gamma_t m_{pk}Q^{p}_{s}Q^{t}_{j}) \\ '\Gamma^{i}_{jk} &= '\overset{m}{\Gamma}{}^{i}_{jk} + \frac{1}{2}\,g^{st}(\gamma_q m_{jk}Q^{q}_{s}Q^{i}_{t} - \gamma_k m_{pj}Q^{p}_{s}Q^{i}_{t} - \gamma_j m_{pk}Q^{p}_{s}Q^{i}_{t}) \end{aligned}$$

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where ${}^{"}\Gamma^{m_i}_{jk}$ and ${}^{'}\Gamma^{m_i}_{jk}$ denote the coefficients of the covariant and contravariant part of the metric general regular connection ($g_{ij,k}=0$, $\gamma_k=0$). These coefficients have the form

$$(1.4)\quad \begin{aligned} {}^{"}\Gamma^{m_i}_{jk} &= \{{}^{i}_{jk}\} + \overset{o}{\nabla}_{[a}P^{i}_{k]}Q^{a}_{j} - \overset{o}{\nabla}_{[a}P^{\ell}_{k]}Q^{ai}g_{j\ell} - \overset{o}{\nabla}_{[a}P^{\ell}_{q]}Q^{ai}P_{k\ell}Q^{q}_{j} \\ {}^{'}\Gamma^{m_i}_{jk} &= \{{}^{i}_{jk}\} + \nabla_{(k}P^{a}_{j)}Q^{i}_{a} - \overset{o}{\nabla}_{[a}P^{\ell}_{k]}Q^{at}Q^{i}_{t}P_{j\ell} - \overset{o}{\nabla}_{[a}P^{\ell}_{j]}Q^{at}Q^{i}_{t}P_{k\ell} \end{aligned}$$

The Weyl-Ōtsuki space of the second kind is defined as a metric Ōtsuki space, which satisfies the conditions a′), b′) and c′) for $m_{ij}=P_{ij}$.

2. In this paper we define the Weyl-Ōtsuki spaces of the second kind as metric Ōtsuki spaces which satisfy the conditions a′), b′) and c) and $m_{ij}=\overset{n}{P}_{ij}=P_{ia_1}P^{a_1}_{a_2}\ldots P^{a_{n-1}}_{j}$. Such an Ōtsuki space is a Weyl-Ōtsuki space of the second kind and the order n. In particular, if $m_{ij}=P_{ij}$, it is a Weyl-Ōtsuki space of the second kind and order 1. Now, we put

$$(2.1)\qquad P^{i}_{j} = S(x)\delta^{i}_{j}$$

where $S(x)$ is a real C^r $(r\geq 3)$ non-zero function: at least one of the second partial derivatives of function $S(x)$ is different from 0.

Taking into account (1.2), we can obtain

$$(2.2)\qquad {}^{'}\Gamma^{i}_{jk} = {}^{"}\Gamma^{i}_{jk} + \frac{S_k}{S}\delta^{i}_{j} \quad \left(S_k = \frac{\partial S(x)}{\partial u^k}\right).$$

We denote a Weyl-Ōtsuki space of the second kind and order n with tensor P in the form (2.1) with SW-O_n.

It is easy to see that connections ${}^{'}\Gamma$ and ${}^{"}\Gamma$ in SW-O_n have the same curvature tensor, named "the curvature tensor of SW-O_n".

3. If the general regular connection in SW-O_n is metric, we can get, using (1.4),

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$$
\begin{aligned}
{}''\Gamma^{m}_{jk}{}^{i} &= \{{}^{i}_{jk}\} + \frac{1}{S}\,(S_j\delta^{i}_{k} - S^{i}g_{jk}) \\
{}'\Gamma^{m}_{jk}{}^{i} &= \{{}^{i}_{jk}\} + \frac{1}{S}\,(S_j\delta^{i}_{k} + S_k\delta^{i}_{j} - Sg^{i}_{jk}),
\end{aligned}
\tag{3.1}
$$

To calculate the curvature tensor of SW-O_n with the metric general regular connection, we put

$$
{}''\Gamma^{m}_{jk}{}^{i} = \{{}^{i}_{jk}\} + H^{i}_{jk}
$$

where H^{i}_{jk} is $\lambda_j\delta^{i}_{k} - \lambda^{i}g_{jk}$ and λ_j stands for $\frac{S_i}{S}$. We obtain, from the well-known formula for the coefficients of the covariant curvature tensor in SW-O_n:

$$
{}''R^{m}_{irkj} = {}'R^{m}_{irkj} = R^{m}_{irkj} =
$$

$$
= K_{irkj} + g_{rk}\psi_{ji} - g_{rj}\psi_{ki} + g_{ij}\psi_{kr} - g_{ik}\psi_{jr} \tag{3.2}
$$

where ψ_{ki} denotes $\overset{o}{\nabla}_k\lambda_i - \lambda_i\lambda_k + 1/2\lambda^s\lambda_s g_{ik}$ and λ_i stands for $\frac{S_i}{S}$. K_{irkj} denotes the corresponding component of the Riemann-Christoffel curvature tensor, with respect to Christoffel symbols.

We can see that all ψ_{ki} are symmetric. Hence, we can easily get the next relations for components of the curvature tensor in SW-O_n:

$$
R^{m}_{irkj} = -R^{m}_{irkj}\,;\; -R^{m}_{rijk}\,;\; R^{m}_{irkj} = R^{m}_{kjir} \tag{3.3}
$$

Transvecting (3.2) by g^{ij}, we obtain

$$
R^{m}_{rk} = K_{rk} + (n-2)\psi_{kr} + g_{kr}\psi_{ji}g^{ji} \tag{3.4}
$$

where R^{m}_{rk} denotes $R^{m}_{irkj}g^{ij}$ and K_{rk} is the corresponding component of the Ricci tensor with respect to Christoffel symbols.

Now, we have to calculate $\psi_{ji}g^{ji}$. We transvect (3.4) by g^{rk} and get

$$
R^{m} = K + 2(n-1)\psi_{ji}g^{ji}
$$

$$
\psi_{ji}g^{ji} = \frac{R^{m} - K}{2(n-1)} \tag{3.5}
$$

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where $\overset{m}{R}$ denotes $\overset{m}{R}_{rk}g^{rk}$ and K is $K_{rk}g^{rk}$. Substituting (3.5) in (3.4), we get

$$\overset{m}{R}_{rk} = K_{rk} + (n-2)\psi_{kr} + g_{kr}\frac{\overset{m}{R} - K}{2(n-1)}$$

$$\psi_{kr} = \frac{K_{rk} - \overset{m}{R}_{rk}}{2-n} + \frac{K - \overset{m}{R}}{2(1-n)(2-n)} \tag{3.6}$$

Taking into account (3.6), we can express (3.2) as follows

$$\overset{m}{R}_{irkj} + \frac{1}{2-n}(g_{ij}\overset{m}{R}_{rk} - g_{ik}\overset{m}{R}_{rj} - g_{rj}\overset{m}{R}_{ik} + g_{rk}\overset{m}{R}_{ij}) +$$

$$+ \frac{\overset{m}{R}}{(n-1)(n-2)}(g_{ij}g_{kr} - g_{ik}g_{rj}) = \tag{3.7}$$

$$= K_{irkj} + \frac{1}{2-n}(g_{ij}K_{rk} - g_{ik}K_{rj} - g_{rj}K_{ik} + g_{rk}K_{ij}) +$$

$$+ \frac{K}{(n-1)(n-2)}(g_{ij}g_{kr} - g_{ik}g_{rj})$$

For every SW-O_n space, there exists a uniquely determined Riemannian space with the same metric tensor. We call such a Riemannian space an adjoint Riemannian space for the space SW-O_n.

So, we have proved the following.

THEOREM 1. *In the SW-O_n space with the metric regular general connection, the tensor on the left-hand side of (3.7) is equal to the conformal curvature tensor of the adjoint Riemannian space.*

COROLLARY 1. *If the adjoint Riemannian space of SW O_n is conformally Euclidean, the tensor on the left-hand side of (3.7) is equal to 0.*

4. If γ_k is different from zero and if SW-O_n is of the order t, we obtain, using (1.3)

$$"\Gamma^i_{jk} = \overset{m}{"\Gamma}{}^i_{jk} + \frac{s^{t-2}}{2}(\gamma^i g_{jk} - \gamma_k\delta^i_j - \gamma_j\delta^i_k)$$

$$'\Gamma^i_{jk} = \overset{m}{'\Gamma}{}^i_{jk} + \frac{s^{t-2}}{2}(\gamma^i g_{jk} - \gamma_k\delta^i_j - \gamma_j\delta^i_k) \tag{4.1}$$

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The connection $'\Gamma$ is semi-symmetric, but $''\Gamma$ is symmetric and these two connections have the same curvature tensor. Thus, we have

$$'\Gamma^i_{jk} = \overset{m}{''\Gamma}{}^i_{jk} + H^i_{jk}$$

$$H^i_{jk} = \lambda^i g_{jk} - \lambda_k \delta^i_j - \lambda_j \delta^i_k$$

$$\lambda_k = \frac{S^{t-2}}{2}\,\gamma_k .$$

It is easy to prove this relation for the curvature tensor

$$R^i_{rkj} = \overset{m}{R}{}^i_{rkj} + \overset{m}{\nabla}_k H^i_{rj} - \overset{m}{\nabla}_j H^i_{rk} + H^s_{rj} H^i_{sk} - H^s_{rk} H^i_{sj}$$

where $\overset{m}{\nabla}$ denotes the classical covariant differentiation with respect to the coefficients of connection $\overset{m}{''\Gamma}$. The metric tensor is recurrent respect to $\overset{m}{\nabla}$ ($\overset{m}{\nabla}_k g_{ij} = \Pi_k g_{ij}$, where Π_k stands for $\frac{-2S_k}{S}$).

For the covariant components of the curvature tensor, we get

$$\begin{aligned} R_{irkj} = {} & \overset{m}{R}_{irkj} + g_{rj}(\lambda_i\lambda_k - \lambda_i\Pi_k + \overset{m}{\nabla}_k\lambda_i) + \\ & + g_{ik}(\lambda_r\lambda_j + \overset{m}{\nabla}_j\lambda_r) - g_{rk}(\lambda_i\lambda_j - \lambda_i\Pi_j + \overset{m}{\nabla}_j\lambda_i) - \\ & - g_{ij}(\lambda_r\lambda_k + \overset{m}{\nabla}_k\lambda_r) - g_{ir}(\overset{m}{\nabla}_k\lambda_j - \overset{m}{\nabla}_j\lambda_k) + \\ & + \lambda^s\lambda_s(g_{rk}g_{ij} - g_{rk}g_{ik}). \end{aligned} \tag{4.2}$$

For the sum $R_{irkj} + R_{rijk}$, we can easily get

$$R_{irkj} + R_{rijk} = \overset{m}{R}_{irkj} + \overset{m}{R}_{rijk} + g_{rj}\theta_{ik} - g_{ij}\theta_{rk} + g_{ik}\theta_{rj} - g_{rk}\theta_{ij} \tag{4.3}$$

where

$$\theta_{ij} = 2\lambda_i\lambda_j - \lambda_i\Pi_j + 2\overset{m}{\nabla}_j\lambda_i - \lambda^s\lambda_s g_{ij} .$$

If $\gamma_k \neq 0$, the components of the curvature tensor do not satisfy the relations analogous to (3.3).

Then, we transvect (4.3) by g^{ij} and we obtain

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$$(4.4)\qquad A_{rk} = \overset{m}{A}_{rk} + (2-n)\theta_{rk} - g_{rk}\theta_{ij}g^{ij}$$

where A_{rk}, $\overset{m}{A}_{rk}$ stand for $(R_{irkj} + R_{rijk})g^{ij}$ and $(\overset{m}{R}_{irkj} + \overset{m}{R}_{rijk})g^{ij}$, respectively. Next, we transvect (4.4) by g^{rk} and we get

$$(4.5)\qquad B = \overset{m}{B} + 2(n-1)\theta_{rk}g^{rk}$$

where B, $\overset{m}{B}$ stand for $A_{rk}g^{rk}$, $\overset{m}{A}_{rk}g^{rk}$, respectively. Hence, from (4.5) we obtain

$$(4.6)\qquad \theta_{rk}g^{rk} = \frac{B - \overset{m}{B}}{2(1-n)}$$

and, substituting (4.6) in (4.4),

$$(4.7)\qquad \theta_{rk} = \frac{A_{rk} - \overset{m}{A}_{rk}}{2-n} + \frac{B - \overset{m}{B}}{2(n-1)(n-2)}\, g_{rk}\,.$$

Substituting (4.7) into (4.3), we obtain

$$\begin{aligned} &R_{irkj} + R_{rijk} + \frac{1}{2-n}\,(g_{ij}A_{rk} - g_{ik}A_{rj} - g_{rj}A_{ik} + g_{rk}A_{ij}) \\ &\qquad + \frac{B}{(n-1)(n-2)}\,(g_{ij}g_{kr} - g_{ik}g_{rj}) = \\ &= \overset{m}{R}_{irkj} + \overset{m}{R}_{rijk} + \frac{1}{2-n}(g_{ij}\overset{m}{A}_{rk} - g_{ik}\overset{m}{A}_{rj} - g_{rj}\overset{m}{A}_{ik} + g_{kr}\overset{m}{A}_{ij}) \\ &\qquad + \frac{\overset{m}{B}}{(n-1)(n-2)}\,(g_{ij}g_{kr} - g_{ik}g_{rj}). \end{aligned} \tag{4.8}$$

Taking into account (3.3), we obtain the following relations

$$\overset{m}{R}_{irkj} = \overset{m}{R}_{rijk}\,;\quad \overset{m}{A}_{rk} = 2\overset{m}{R}_{rk},\quad \overset{m}{B} = 2\overset{m}{R}\,.$$

We can see that the tensor on the right-hand side of (4.8) is equal to $2C_{irkj}$ where C_{irkj} denotes the corresponding component of Weyl´s conformal curvature tensor of the adjoint Riemannian space.

We have proved

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THEOREM 2. *In SW-O_n with $\gamma_k \neq 0$, the tensor on the left-hand side of (4.8) is equal to* $2C_{irkj}$.

COROLLARY 2. *If the adjoint Riemannian space is conformally Euclidean, the tensor on the left-hand side of (4.8) in SW-O_n is equal to 0.*

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REZIME

WEYL-ŌTSUKI-JEVI PROSTORI DRUGE VRSTE SA SPECIJALNIM TENZOROM P

U ovom radu, definisani su Weyl-Ōtsuki-jevi prostori druge vrste reda n (u oznaci SW-O_n) sa specijalnim tenzorom P oblika $P^i_j = S(x)\delta^i_j$ gde je $S(x)$ C^r funkcija (nenula i nelinearna). Ovo je prvi stepen uopštenja od Rimanovih prostora ka metričkim Ōtsuki-jevim prostorima sa regularnom opštom koneksijom. Ovde su uspostavljene veze izmedju tenzora krivine i tenzora konforne krivine u pridruženom Rimanovom prostoru i odgovarajućih objekata u SW-O_n.

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О ОДНОМ КЛАССЕ КОНЕЧНЫХ к-ПОЛУСЕТЕЙ

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РЭЗЮМЕ

к-Полусети |1| являются одним обобщением к-сетей |4-5|. В настоящей работе рассматривается один класс конечных к-полусетей и получается связь конечных аффиных пространств Спернера и к-полусетей принадлежащих одному подклассу рассматриваемого класса к-полусетей.

к-Полусети, описанные автором в |1|, являются одним обобщением к-сетей |4-5|. Как известно |4-5|, каждой к-сети соответствует ортогональная система квазигрупп, и наоборот. В |1| получена характеризация к-полусетей через ортогональные системы частичных квазигрупп (ОСЧК), принадлежащих одному подклассу класса ОСЧК. Конечные аффинные плоскости характеризованны через к-сети специального рода. Аффинные пространства Спернера (АПС) описанны Спернером в |2| (|3|, стр. 293-294). В настоящей работе рассматривается один класс к-полусетей и получается связь конечных аффинных пространств Спернера и к-полусетей принадлежащих одному подклассу рассматриваемого класса к-полусетей

ОПРЕДЕЛЕНИЕ 1. |1| Пусть T непустое множество и пусть непустое множество L множество некоторых подмножеств множества T. Множества $L_1,\dots,L_k$, $k \in N \setminus \{1,2\}$, пусть разбивают множество L. Элементы множества L называются точками, элемент множества L прямыми. Множества $L_1,\dots,L_k$ называются классами прямых $(T, L_1,\dots,L_k)$ называется к-полусетью тогда и только тогда когда выполняются следующие условия.

М1. Пересечение каждых двух прямых принадлежащих различным классам L_i, L_j, $i,j \in \{1,\dots,k\}$, является одноэлементным

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множеством или пустим множеством [1];

M2. Каждая точка из T находится в одной и только в одной прямой каждого класса L_i, $i \in \{1,\dots,k\}$.

ПРИМЕЧАНИЕ 1. 1.1. Учитывая M2, получаем что прямые принадлежащие одному и тому же классу L_i, $i \in \{1,\dots,k\}$, не пересекаютея, т.е. что их пересечение является пустым множеством.

1.2. Учитывая $k \in N \setminus \{1,2\}$ и M2, находим, что $|T| \geq 3$.

В настоящей работе рассматриваются только конечные k-полусети, т.е. k-полусети $(T, L_1, \dots, L_k)$ в которых T является конечным множеством.

$\mathrm{Max}\{|\ell| \mid \ell \in L\}$ называется T-порядок, а $\mathrm{Max}\{|L_i| \; i \in \{1,\dots,k\}\}$ L-порядок k-полусети $(T, L_1, \dots, L_k)$ | 1 |; $L = L_1 \cup \dots \cup L_k$. Справедливо:

ЛЕММА 1. | 1 | T-порядок $\leq$ L-порядок.

Если в определении 1 вместо 1 берется

M1. Пересечение каждых двух прямых принадлежащих различным классам L_i, L_j, $i,j \in \{1,\dots,k\}$, является одноэлементным множеством [2]

то $(T, L_1, \dots, L_k)$ станет k-сетью.

В настоящей работе рассматриваетея класс k-полусетей удовлетворяющих следующему условию:

M3. $(\forall \ell \in L)\,|\ell| = m \in N$

ПРИМЕЧАНИЕ 2. Учитывая $k \in N \setminus \{1,2\}$ и M2, находим, что $m \in N \setminus \{1\}$. Отсюда, учитывая лемму 1, находим, что q = L-порядок $\in N \setminus \{1\}$

УТВЕРЖДЕНИЕ 2. Пусть T непустое множество и пусть непустое множество L множество некоторых подмножеств множества T. Множества $L_1, \dots, L_k$, $k \in N \setminus \{1,2\}$, пусть разбывают множество L.

---

1) Удобнее: две прямые различных классов пересекаются не больше чем в одной точке.

2) Удобнее: две прямые различных классов пересекаются в одной и только в одной точке.

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Как в определении 1, пусть, элементы множества T называются точками, элементы множества L прямыми, а множества $L_1,\ldots,L_k$ классами прямых. Тогда имеет место:

1^o Если система $(T,L_1,\ldots,L_k)$ удовлетворяет условиями М2 и М3, то $(T,L_1,\ldots,L_k)$ удовлетворяет и следующему условию:

М4. $|L_i| = |L_j| = q \in N\setminus\{1\}$ для любых $i,j\in\{1,\ldots,k\}$;

2^o Существует система $(T,L_1,\ldots,L_k)$ удовлетворяющая условиям М2 и М3 и не удовлетворяюща условию М1;

3^o Сущецтвует к-полусеть $(T,L_1,\ldots,L_k)$ удовлетвор - яющая М4 и не удовлетворяющая М3;

4^o Если к-полусеть $(T,L_1,\ldots,L_k)$ удовлетворяет М3, то $m \le q$; и

5^o Если к-полусеть $(T,L_1,\ldots,L_k)$ удовлетворяет условиям М3 и $m=q$, то $(T,L_1,\ldots,L_k)$ являетса к-сетью.

ДОКАЗАТЕЛЬСТВО. Учитывая М2 и М3, находим, что имеет место следующие равенства

$$|T| = |L_i|\cdot m = |L_j|\cdot m$$

для любых $i,j \in \{1,\ldots,k\}$. Следовательно, имеет место М4.

Учитывая систему $(T,L_1,\ldots,L_k)$ на Рис. 1 (так же на Рис. 2, где $m=q$), находим, что имеет место 2^o.

Учитывая 3-полусеть на Рис. 3, находим, что имеет место 3^o.

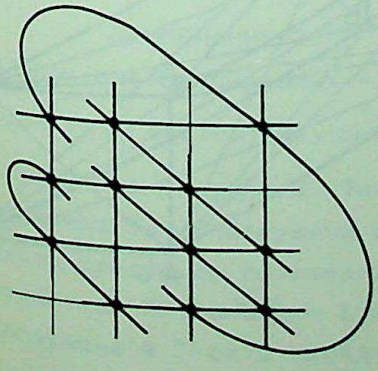

Рис. 1

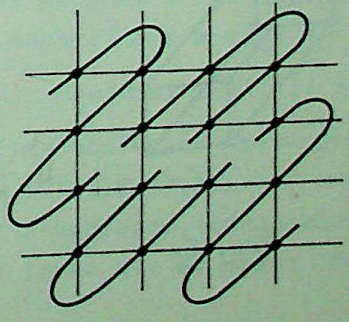

Рис. 2

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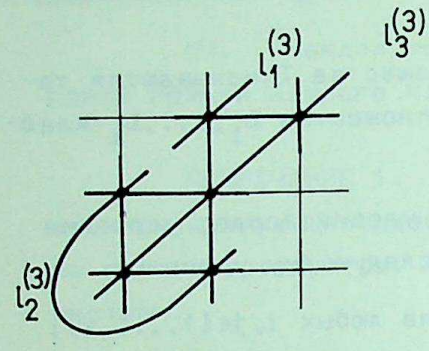

Рис. 3

В силу леммы 1 и 1^o, так как m = T-порядок и q = L-порядок находим, что имеет место 4^o.

Учитывая M1, M2 и факт что $|L_i| \in N$, находим, что имеет место 5^o.

Примеры k-полусетей удовлетворяющие M3 изображены на рисунках 4-8. 4-Полусеть на рисунке 4 является 4-сетю . На рисунках 5_1 и 5_2 изображена одна и та же 3-полусеть .

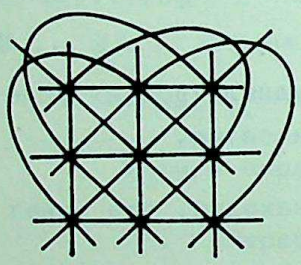

Рис. 4

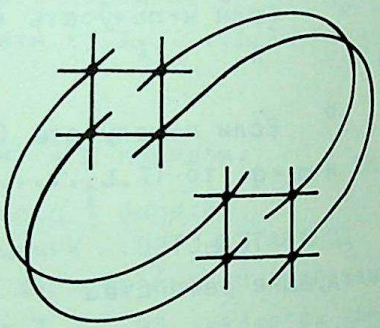

Рис. 5_1

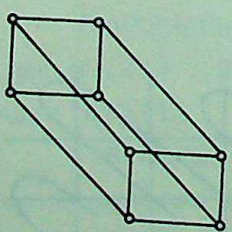

Рис. 5_2

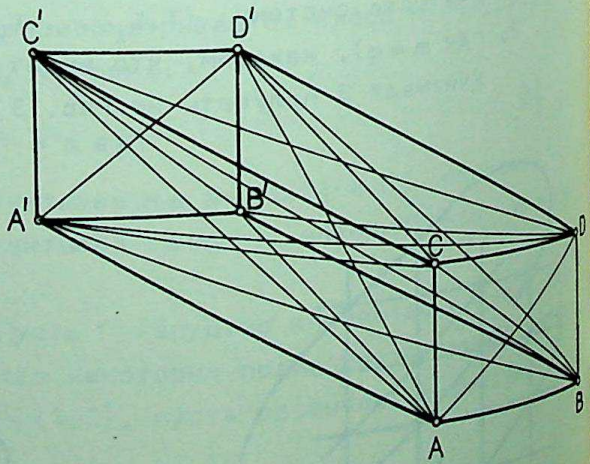

Рис. 6

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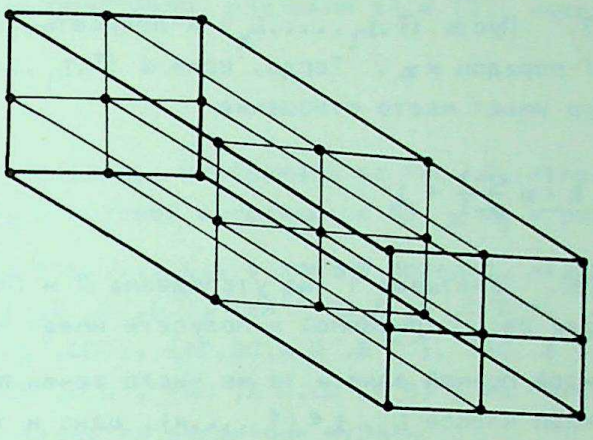

Рис. 7

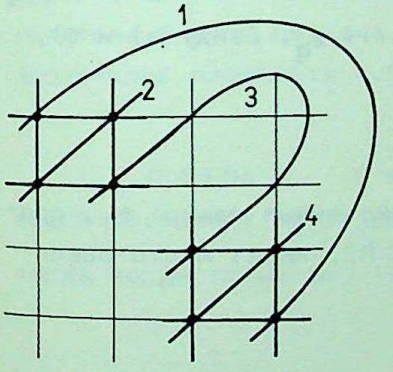

Рис. 8

	1	2	3	4
1	1	2		
2	2	3		
3			3	4
4			4	1

Рис. 9

ПРИМЕЧАНИЕ 3. 3-Полусеть на рисунке 8 принадлежит классу рассматривармых к-полусетей с параметрами $m = 2$ и $q = 4$. Одна из соответствующих частичных квазигрупп $(|1|)$ изображена на рисунке 9. Так как упомянутая частичная квазигруппа не может быть погружена в квазигруппы $(\{1,2,3,4\},A)$, то имеет место следующее положение: существует 3-полусеть, обладающая свайством М3, имеющая L-парядок $= q \in N$ такая что не может быть погружена в 3-сеть порядка q .

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ТЕОРЕМА 3. Пусть $(T, L_1, \ldots, L_k)$ к-полусеть, имеющая L-порядок $= q$ и T-порядок $= m$. Тогда, если в $(T, L_1, \ldots, L_k)$ выполняется М3, то имеет место отношение

$$k \leq m \frac{q-1}{m-1} + 1 \qquad ^{1)}$$

ДОКАЗАТЕЛЬСТВО. Учитывая 1^{o} из утверждения 2 и Примечание 2, находим, что для рассматриваемой к-полусети имеет место:

а) В каждой прямой одно и то же число точек $m \in N \setminus \{1\}$;

б) В каждом классе L_i, $i \in \{1, \ldots, k\}$, одно и то же число прямых $q \in N \setminus \{1\}$.

Пусть $T \in T$ в $\ell_1 \in L_i$. Пусть, далее, $\ell_2, \ldots, \ell_q \in L_i \setminus \{\ell_1\}$. Учитывая М2, находим, что каждая из точек множества $\ell_2 \cup \ldots \cup \ell_q$ в одной и только в одной из прямых $\ell_2, \ldots, \ell_q$. Ввиду а) и б), следовательно, имеет место равенство

$$(1) \qquad |\ell_2 \cup \ldots \cup \ell_q| = m(q-1).$$

Если $\bar{k}$ число всех прямых, за исключением прямой ℓ_1, проходящих через точку T, то, на основании М2, имеет место равенство

$$(2) \qquad k = \bar{k} + 1.$$

Каждая из $\bar{k}$ упоманнутых прямых $\bar{\ell}_1, \ldots, \bar{\ell}_{\bar{k}}$, за исключением точки T, содржит $m-1$ точку. Так как $\bar{\ell}_1 \cap \ldots \cap \bar{\ell}_{\bar{k}} = \{T\}$, учитывая М1, находим, что имеет место равенство

$$(3) \qquad |(\bar{\ell}_1 \cup \ldots \cup \bar{\ell}_{\bar{k}}) \setminus \{T\}| = \bar{k}(m-1).$$

Так как $\ell_1 \cap \bar{\ell}_1 \cap \ldots \cap \bar{\ell}_{\bar{k}} = \{T\}$ (М1), $\ell_1 \cap (\ell_2 \cup \ldots \cup \ell_q) = \emptyset$ (М2) и $\ell_1 \cup \ldots \cup \ell_q = T$ (М2), то учитывая (1) и (3), находим, что имеет место следующее отношение:

$$(4) \qquad \bar{k}(m-1) \leq m(q-1) .$$

---

1) Если $m = q$, то $k \leq m\frac{q-1}{m-1}+1$ превращается в $k \leq q+1$ -известное отношение, справедливо в к-сетях $(|g| > 1)$, см. утверждения 2).

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Следовательно, учитывая а) и (2), находим, что теорема доказана.

7-Полусеть, изображена на Рис. 6, является примером к-полусетей, в которой выполняются М3, $m < q$ и равенство $к = m\frac{q-1}{m-1}+1$. (Множества $T, L_1, \ldots, L_7$, в том же проядке, являютса: {A,B,C,D,A´,B´,C´,D´}, {AB[1],CD,A´,B´,C´D´}, {AC,BD,A´C´,B´D´}, {AA´,BB´,CC´,DD´}, {AD,BC,A´D´,B´C´}, {AC´,A´C ,BD´,B´D} , {AB´,A´B,CD´,C´D}, {AD´,A´D,CB´,C´B} ; на Рис.6, A,B,C,D,A´, B´,C´,D´ являются вершинами куба).

ПРИМЕЧАНИЕ 4. Только что рассматриваемая 7-полусеть построена, как уже упомянуто, на множестве T точек, являющихся вершинами куба (Рис. 6). Для построения этой 7-полусети использована 3-сеть парядка 2, которую "полагали" в стороны и диагональные плоскости куба.

ТЕОРЕМА 4. В к-полусети $(T, L_1, \ldots, L_k)$, удовлетворяющей М3, имеет место равенство $к = m\frac{q-1}{m-1} + 1$, тогда и только тогда когда любые две точки коллинеарны.

ДОКАЗАТЕЛЬСТВО. 1) $\Longrightarrow$ (справедливо: $к = m\frac{q-1}{m-1} + 1$)

Пусть A и B , $A \neq B$, любые точки из T . В к-полусетях (как и в к-сетях), вообще говоря, не всякие две точки коллинеарны. Существует не больше чем один класс L_i такой что через A и B, $A \neq B$, проходит одна прямая из L_i(M1). Пусть L_i не является таким классом. (Ввиду $k \in N \setminus \{1,2\}$, такой класс существует.) Пусть, далее $A \in \ell_1 \in L_i$, $B \in \ell_2 \in L_i$ и $\ell_1 \neq \ell_2$; $L_i = \{\ell_1, \ell_2, \ldots, \ell_q\}$ (M2 и 1^o из утверждения 2). $m(q-1)$ является числом точек, принадлежащих прямаиым $\ell_2, \ldots, \ell_q$ (Прим. 1 и М3). Учитывая М3, М1 и факт что $A \not\in \ell_2, \ldots, \ell_q$, находим, что в прямых, проходящих через

1) {X,Y} обозначаетс через XY.

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точку A, за исключением прямой ℓ_1 (в k-1 прямах), находится (k-1)(m-1) точек, принадлежащих сразу прямым $\ell_2,\ldots,\ell_q$. Отсюда, предполагая что A и B, $A \neq B$, не являются коллинеарными, следует что

$$(k-1)(m-1) < m(q-1)$$

т.е. что

$$k < m\,\frac{q-1}{m-1} + 1.$$

Следовательно, любые две точки коллинеарны.

2) $\Longleftarrow$ (справедливо: любые две точки коллинеарны)

Пусть A любая точка из T. Пусть, далее, $A \in \ell_1 \in L_i$ (ввиду M2, любая точка из T в некоторой прямой). Отсюда, учитывая примечание 2, 1^o из утверждения 2 и примечание 1, находим, что существует в точности $q-1 \in N$ прямых $\ell_2,\ldots,\ell_q \in L_i$ таких что $A \notin \ell_2,\ldots,\ell_q$.

Из M2 следует, что $\ell_1 \cup \ell_2 \cup \ldots \cup \ell_q = T$. Учитывая примечание 1 и 1^o из утверждения 2, находим, что $|\ell_2 \cup \ldots \cup \ell_q| = (q-1)m$,.

Через точку A проходит k-1 прямых, за исключением прямой ℓ_1. В каждой из этих прямых находится m-1 точек, отличающихся от A (M3). Таким образом, в этих прямых находится в точности (k-1)(m-1) точек, отличающихся от A и коллинеарных с точкой A. Кроме этых точек, за исключением точек прямой ℓ_1, коллинеарных с точкой A не существует; ввиду M1, точки из того множества не являются сразу точками прямой ℓ_1. Отсюда, так как A коллинеарна с любой из точек принадлежащих прямам $\ell_2,\ldots,\ell_q$, находим, что имеет место:

$$\neg((k-1)(m-1) < m(q-1))$$

т.е.

$$(k-1)(m-1) \geq m(q-1)$$

т.е.

$$k \geq m\,\frac{q-1}{m-1} + 1\ .$$

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Отсюда, учитывая Теорему 3, находим, что

$$k = m\frac{q-1}{m-1} + 1$$

В к-сетях не пересекаются только прямые, принадлежащие одному и тому же классу (М1). Прямые p_1 и p_2, принадлежащие одному и тому же классу, называются параллельными. Это отношение является РСТ отношением на множестве $L = L_1 \cup \ldots \cup L_k$. В к-полусетях, не являющихся к-сетями, существуют прямые p_1 и p_2, $p_1 \neq p_2$, не принадлежащие одному и тому же классу, такие что $p_1 \cap p_2 = \emptyset$. В к-полусетях, не являющихся к-сетями, не является справедливым положение: для любых $\ell_1, \ell_2, \ell_3 \in L = L_1 \cup \ldots \cup L_k$ из $\ell_1 \cap \ell_2 = \emptyset$ и $\ell_2 \cap \ell_3 = \emptyset$ следует $\ell_1 \cap \ell_3 = \emptyset$ (см., на пример, Рис. 6). Но, если речь идет о прямых ℓ_1, ℓ_2, ℓ_3, принадлежащих одному и тому же классу, то, только что упомянутое, положение справедливо, как и у к-сетях. Таким образом, вообще говоря, в к-полусетях, кроме отношения "параллельности прямых", может быть непустым и отношение "скрещивания прямых" на $L = L_1 \cup \ldots \cup L_k$.

ОПРЕДЕЛЕНИЕ 2_1. Пусть $(T, L_1, \ldots, L_k)$ любая к-полусеть. $p_1 \in L$ и $p_2 \in L$, скажем, параллельние тогда и только тогда когда являются прямими одного и того же класс L_i, $i \in \{1, \ldots, к\}$ $(p_1 \| p_2 \overset{\text{def}}{\Longleftrightarrow} p_1, p_2 \in L_i)$.

ОПРЕДЕЛЕНИЕ 2_2. Пусть $(T, L_1, \ldots, L_k)$ любая к-полусеть. $p_1 \in L$ и $p_2 \in L$, скажем, в отношении скрещивания тогда и только тогда когда $p_1 \cap p_2 = \emptyset$ и $p_1 \nparallel p_2$.

ЛЕММА 5. Пусть $(T, L_1, \ldots, L_k)$ к-полусеть, удовлетворяющая условиям М3 и $k = m\frac{q-1}{m-1} + 1$. Тогда имеет место:

а) Если A и B, $A \neq B$, любые точки из T, то существует одна и только одна прямая p такая что $A \in p$ и $B \in p$;

б) Все прямые имеют одно и то же число точек;

в) Отношение параллельности является РСТ отношением на множестве $L = L_1 \cup \ldots \cup L_k$;

г) $(\forall A \in T)(\forall p \in L)(\exists! p' \in L)(p' \| p \wedge A \in p')$

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д) Пусть $A \in T$, $p \in L$ и $A \notin p$. Тогда существует в точности $k-(m+1)$ прямых, проходящих через точку A, являющихся в отношении скрещивания с p [1)]

ДОКАЗАТЕЛЬСТВО $\bar{а}$) Учитывая Теорему 4, находим, что любые $A,B \in T$, $A \neq B$, являются коллинеарными. Отсюда, ввиду M1, получаем, что имеет место а).

$\bar{б}$) Положение б) является условием M3.

$\bar{в}$) Множество $\{L_1,\dots,L_k\}$ является разбиением множества $L = L_1 \cup \dots \cup L_k$ (Определение 1). Отсюда, на основании Определения 2_1, находим, что имеет место в).

$\bar{г}$) Пусть $\ell \in L_i$. Ввиду M2, через каждую точку A проходит одна и только одна $\ell' \in L_i$. Отсюда, учитывия определение 2_1, получаем. что имеет место г).

$\bar{д}$) Пусть $\ell \in L$ и $A \notin \ell$ (см. примечание 2). Через A проходит k и только k прямых (из каждого класса в точности одна- M2). На основании г), одна из этих прямых паралельна с ℓ. На основаним Теоремы 4, m прямых пересекается с прямой ℓ. Следовательно, положение д) имеет место.

k-Полусети $(T,L_1,\dots,L_k)$, в которых имеет место M3 и равенство $k=m\frac{q-1}{m-1}+1$, находятся в самой близкой связи с аффинными пространствами Спернера (АПС). (Если $m=q$, то речь идет о k-сетях, в которых имеет место равенство $k=q+1$, и они характеризуют аффинные плоскоси; 5^o из Утверждения 2, |4-5|).

ОПРЕДЕЛЕНИЕ 3. |2| [2)] Пусть T непустое конечное множество, а непустое множество L пусть множество некоторых подмножеств множества T. Элементи множества T называются точками, а элементи множества L прямыми. (T,L) называется аффинным пространством Спернера (АПС), тогда и только тогда когда справедливо:

S1. Если A и B, $A \neq B$, любые точки из T, то существует одна и только одна прямая p такая что $A \in p$ и $B \in p$;

---

1) Если $m=q$, то $(T,L_1,\dots,L_k)$ k-сеть (5^o из утверждения 2), в которой справедливо $k=q+1$, и тогда $k-(m+1)=0$.

2) |3|, стр. 293-294.

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S2. Все прямые имеют одно и то же число точек;

S3. Отношение параллельности ($||$) является РСТ отношением на множестве L ; и

S4. $(\forall A \in T)(\forall p \in L)(\exists! p' \in L)(p' || p \wedge A \in p')$ [1]

Учитывая а) - г) из леммы 5 и определение 3, находим, что имеет место:

ТЕОРЕМА 6. Каждой к-полусети $(T, L_1, \ldots, L_k)$, в которой имеет место М3 и равенство $k = m\frac{q-1}{m-1} + 1$, соответствует аффинное пространство Спернера (T, L) где $L = L_1 \cup \ldots \cup L_k$.

Пусть T непустое множество. Система $(T, \{T\})$ удовлетворяет условиям S1 - S4. Такая система обладает только одной прямой. Удобно ее назвать тривиальным АПС.

Число точек в прямых у нетривиальных АПС больше двух или равно двум ($m \geq 2$). Предположение что справедливо противоположное ($m = 1$), именно, ввиду S1, влечет в контрадикцию с S3.

Параллельные прямые в АПС или совпадают или имеют пустое пресечение (S3 - Р, S4). Отсюда ввиду $m \geq 2$, получаем, что у нетривиальных АПС сущецтвуют по меньшей мере три попарно непараллельные прямые.

Отсюда, далее, получаем, что $|L/||| \geq 3$. Положим: $L/|| = \{L_1, \ldots, L_k\}$; $k \in N \setminus \{1,2\}$.

В связи с каждым нетривиальным АПС (T, L), именно, можно рассматривать систему $(L, L_1, \ldots, L_k)$, где $\{L_1, \ldots, L_k\} = L/||$ и $k \in N \setminus \{1,2\}$.

На основании S1, получаем, что в только что построенной системе $(T, L_1, \ldots, L_k)$ справедливо М1. Из S1, именно, следует

$$|p_1 \cap p_2| > 1 \Rightarrow p_1 = p_2$$

что является эквивалентным с

$$p_1 \neq p_2 \Rightarrow |p_1 \cap p_2| \leq 1.$$

Далее, на основании S4, получаем, что в $(T, L_1, \ldots, L_k)$ справедливо М2. Именно: 1^o если $A \in p \in L_i$, то, на основании S4,

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1) В |3| использовано отношение инцидентности.

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p является однозначно определенной прямой, паралельной с p; и 2^{o} если $A \notin p \in L_i$, то, на основании S4, существует одна и только одна p', такая что $p' \| p$ и $A \in p'$.

M3 совпадает с S2.

Из S1 следует, что каждая пара точек в $(T, L_1, \ldots, L_k)$ является коллинеарной .

Отсюда, учитывая Теорему 4, находим, что имеет место:

ТЕОРЕМА 7. Каждому конечному нетривиальному АПС (T,L) соответствует к-полусеть $(T, L_1, \ldots, L_k)$, $\{L_1, \ldots, L_k\} = L/\|$, в которой имеет место M3 и равенство $k = m\,\frac{q-1}{m-1} + 1$.

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REZIME

O JEDNOJ KLASI KONAČNIH k-POLUREŠETAKA

k-Polurešetke (|1|) predstavljaju jedno uopštenje k-rešetaka (|4-5|). Svakoj k-rešetki odgovara ortogonalni sistem kvazigrupa, i obrnuto. U |1| je pokazano da svakoj k-polurešetki odgovara regularno ortogonalni sistem regularnih parcijalnih kvazigrupa, i obrnuto. Specijalne k-rešetke karakterišu konačne afine ravni |4-5|. U ovom radu se razmatra jedna klasa konačnih k-polurešetaka i nalazi veza izmedju jedne njene podklase i afinih prostora Spernera.

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ON BISYMMETRIC [n,m]-GROUPOIDS

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ABSTRACT

In this paper bisymmetric [n,m]-groupoids (defined in |6|) are considered. It is shown that every bisymmetric [n,m]-groupoid Q(f) is commutative, if the range of at least one of its component operations is Q. A corollary of the preceding proposition is that there do not exist proper bisymmetric [n,m]-quasigroups (which was proved in |6|). It is also shown that from the commutativity and mutual mediality of the component operations follows the bisymmetry of an [n,m]-groupoid. A characterization of the components which are n-quasigroups of a bisymmetric [n,m]-groupoid is given. Necessary and sufficient conditions for an [n,m]-groupoid to be bisymmetric with n-quasigroup components are obtained.

First we shall give some basic definitions. The notions from the general theory of n-quasigroups can be found in |1|.

Instead of $x_p, x_{p+1}, \ldots, x_q$ we shall write $\{x_i\}_{i=p}^{q}$ or x_p^q. If $p > q$, then x_p^q will be considered empty.

An n-groupoid Q(f) is called commutative iff the following identity holds

$$f(x_1^n) = f(x_{\phi(1)}^{\phi(n)}) ,$$

for every permutation ϕ of the set $N_n = \{1,\ldots,n\}$.

An n-groupoid Q(f) is totally symmetric iff

$$f(x_1^n) = x_{n+1} \quad \text{implies} \quad f(x_{\psi(1)}^{\psi(n)}) = x_{\psi(n+1)} \quad \text{for all } x_i \in Q ,$$

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$i = 1,\ldots,n+1$ and every permutation ψ of N_{n+1}.

Two n-groupoids $Q(f)$ and $Q(g)$ are mutually medial iff

$$f(\{g(x_{il}^{in})\}_{i=1}^{n}) = g(\{f(x_{lj}^{nj})\}_{j=1}^{n})$$

holds for all $x_{ij} \in Q$, $i,j = 1,\ldots,n$. A n-groupoid which is medial to itself is called medial.

An n-groupoid $Q(f)$ is bisymmetric iff

$$f(\{f(x_{il}^{in})\}_{i=1}^{n})$$

is invariant for every permutation of elements $x_{ij} \in Q$, $i,j = 1,\ldots,n$.

Let Q be a nonempty set, n and m positive integers and $f : Q^n \to Q^m$. Then $Q(f)$ is called an $[n,m]$-groupoid. The n-ary operations $f_1,\ldots,f_m$ defined by

$$f(x_1^n) = (y_1^m) \iff (\forall i \in N_m)\ y_i = f_i(x_1^n)$$

are called the component operations of f and this is denoted by $f = (f_1,\ldots,f_m)$. An $[n,m]$-groupoid is proper iff $n,m, |Q| \geq 2$.

An $[n,m]$-groupoid $Q(f)$ is called an $[n,m]$-quasigroup (or multiquasigroup) ($|2|$, $|3|$) iff for every injection $\phi : N_n \to N_{n+m}$ and every $(a_1^n) \in Q^n$ there exists a unique $(b_1^{n+m}) \in Q^{n+m}$ such that

$$f(b_1^n) = (b_{n+1}^{n+m}) \quad \text{and} \quad b_{\phi(i)} = a_i,\ i = 1,\ldots,n.$$

An $[n,m]$-groupoid $Q(f)$ is commutative iff

$$f(x_1^n) = f(x_{\phi(1)}^{\phi(n)})$$

holds for all $x_i \in Q$, $i = 1,\ldots,n$ and every permutation ϕ of N_n. Obviously, an $[n,m]$-groupoid is commutative iff all its component operations are commutative n-groupoids.

Let $Q(f)$ be an $[n,m]$-groupoid, $f = (f_1,\ldots,f_m)$. If $X = [x_{ij}]$ is an $n \times p$ array of elements from Q, then an $m \times p$ array $C_f(X) = [y_{ij}]$ is defined by $y_{ij} = f_i(x_{lj}^{nj})$, $i = 1,\ldots,m$, $j = 1,\ldots,p$.

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If $X = [x_{ij}]$ is an $r \times n$ array of elements from Q, then an $r \times m$ array $R_f(X) = [z_{ij}]$ is defined by $z_{ij} = f_j(x_{il}^{in})$, $i=1,\ldots,r$, $j=1,\ldots,m$. An $[n,m]$-groupoid $Q(f)$ is said to be bisymmetric (|6|) iff for all $n \times n$ arrays $X = [x_{ij}]$ of elements from Q and every permutation ϕ of $N_n \times N_n$

$$C_f R_f(X) = C_f R_f(X^{\phi}) ,$$

where $X^{\phi} = [x_{\phi(i,j)}]$, holds.

It is obvious that the components of a bisymmetric $[n,m]$-groupoid are bisymmetric. But from the bisymmetry of the components the bisymmetry of an $[n,m]$-groupoid does not follow (|6|). In |6| it is also shown that there are bisymmetric $[n,m]$-groupoids which are not commutative. But this does not hold for a class of bisymmetric $[n,m]$-groupoids which are described in the following proposition.

If $Q(f)$ is an n-groupoid, by Rf we shall denote the range of f.

THEOREM 1. *Let $Q(f)$ be a bisymmetric $[n,m]$-groupoid, $f = (f_1,\ldots,f_m)$, such that there is a component operation f_k for which $Rf_k = Q$. Then each of the component operations f_i, $i = 1,\ldots,m$, is commutative.*

P r o o f. $Q(f)$ is a bisymmetric $[n,m]$-groupoid, so for every every $\ell \in N_m$ and any permutation ϕ of $N_n \times N_n$, the following identity holds

$$f_\ell(\{f_k(x_{il}^{in})\}_{i=1}^{n}) = f_\ell(\{f_k(x_{\phi(i,1)}^{\phi(i,n)})\}_{i=1}^{n}) .$$

If a_i, $i = 1,\ldots,n$ are arbitrary elements from Q, then, since $Rf_k = Q$, there exist elements $b_{ij} \in Q$, $i,j = 1,\ldots,n$ such that

$$f_k(b_{il}^{in}) = a_i, \quad i = 1,\ldots,n .$$

If ϕ is an arbitrary permutation of the set N_n and ψ a permutation of $N_n \times N_n$ such that $\psi(p,q) = (\phi(p),q)$ for all $p,q \in N_n$, then

$$f_\ell(a_1^n) = f_\ell(\{f_k(b_{\phi(i),1}^{\phi(i),n})\}_{i=1}^{n} = = f_\ell(a_{\phi(1)}^{\phi(n)}) .$$

Hence, f_ℓ is a commutative n-groupoid for all $\ell \in N_m$.

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COROLLARY 1.1. *Every bisymmetric* $[n,m]$*-groupoid* $Q(f), f=(f_1,\ldots,f_m)$ *, such that there is a component operation* f_k *for which* $Rf_k=Q$*, is a commutative* $[n,m]$*-groupoid.*

COROLLARY 1.2. *In* |6| *it is proved that every bisymmetric multiquasigroup is commutative (which implies that there are no proper bisymmetric multiquasigroups because in* |7| *it is proved that there are no proper commutative multiquasigrpups). This is a consequence of Theorem 1 since all components of an* $[n,m]$*-quasigroup* $Q(f)$*,* $f=(f_1,\ldots,f_m)$ *, are* n*-quasigroups, hence* $Rf_k=Q$ *for all* $k\in N_m$.

THEOREM 2. *Let* $Q(f)$ *be an* $[n,m]$*-groupoid,* $f=(f_1,\ldots,f_m)$. *If all component operations* f_i*,* $i=1,\ldots,m$ *are commutative and every two component operations are mutually medial, then* $Q(f)$ *is a bisymmetric* $[n,m]$*-groupoid.*

P r o o f. If f_k and f_ℓ are two commutative n-groupoids which are mutually medial, then it can be proved (by a similar argument as it is done in |5|, where the special case $f_\ell=f_k$ was considered) that

$$f_k(\{f_\ell(x_{il}^{in})\}_{i=1}^{n}) = f_k(\{f_\ell(x_{\phi(i,1)}^{\phi(i,n)})\}_{i=1}^{n})$$

holds for all $x_{ij}\in Q$, $i,j=1,\ldots,n$ and any permutation ϕ of $N_n\times N_n$. Hence, $Q(f)$ is a bisymmetric $[n,m]$-groupoid.

THEOREM 3. *Let* $Q(f)$ *be a bisymmetric* $[n,m]$*-groupoid,* $f=(f_1,\ldots,f_m)$*. If* f_k *is an* n*- quasigroup, then there exist an Abelian group* $Q(+)$ *and an element* $a\in Q$ *such that*

$$f_k(x_1^n) = \alpha\sum_{i=1}^{n} x_i + a\ ,$$

where α *is an automorphism of the group* $+$.

P r o o f. Since $Q(f)$ is bisymmetric it follows that f_k is a bisymmetric n-quasigroup. This means that f_k is a medial n-quasigroup, hence there exist an Abelian group $Q(+)$ and an element $a\in Q$ such that

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$$f_k(x_1^n) = \sum_{i=1}^{n} \alpha_i x_i + a$$

where α_i, $i = 1,\dots,n$ are automorphisms of the group $+$ ($|1|$). From Theorem 1 it follows that f_k is commutative. If in $f_k(x_1^n) = f_k(x_{\phi(1)}^{\phi(n)})$ we put $x_1 = x_{\phi(2)} = x$ and 0 otherwise (where 0 is the neutral element of the group $+$), we get

$$\alpha_1 x + \alpha_2 0 + \dots + \alpha_n 0 + a = \alpha_1 0 + \alpha_2 x + \alpha_3 0 + \dots + \alpha_n 0 + a ,$$

thus $\alpha_1 x = \alpha_2 x$ for all $x \in Q$. By a similar argument it can be shown that $\alpha_1 = \dots = \alpha_n$.

THEOREM 4. *Let* $Q(f)$ *be an* $[n,m]$*-groupoid,* $f=(f_1,\dots,f_m)$. $Q(f)$ *is a bisymmetric* $[n,m]$*-groupoid the components of which are* n*-quasigroups iff there exist an Abelian group* $Q(+)$, *elements* $a_i \in Q$ $i=1,\dots,m$ *and automorphisms* α_i, $i = 1,\dots,m$ *of the group* $+$ *such that*

$$f_i(x_1^n) = \alpha_i \sum_{j=1}^{n} x_j + a_i, \quad i = 1,\dots,m.$$

P r o o f. Let $Q(f)$ be a bisymmetric $[n,m]$-groupoid and let all component operations f_i, $i = 1,\dots,m$ be n-quasigroups. Then by Theorem 3 it follows that there exist Abelian groups $Q(\overset{i}{+})$, $i = 1,\dots,m$, and elements $a_i \in Q$, $i = 1,\dots,m$ such taht

$$f_i(x_1^n) = \alpha_i(x_1 \overset{i}{+} \dots \overset{i}{+} x_n) \overset{i}{+} a_i, \quad i = 1,\dots,m ,$$

where α_i is an automorphism of the group $\overset{i}{+}$, $i = 1,\dots,m$.

Let k,j be arbitrary elements from N_m. The identity

$$f_k(\{f_j(x_{i1}^{in})\}_{i=1}^{n}) = f_k(\{f_j(x_{\phi(i,1)}^{\phi(i,n)})\}_{i=1}^{n}),$$

where ϕ is a permutation of $N_n \times N_n$, then, can be written in the form

$$\alpha_k((\alpha_j(x_{11} \overset{j}{+} \dots \overset{j}{+} x_{1n}) \overset{j}{+} a_j) \overset{k}{+} \dots \overset{k}{+} (\alpha_j(x_{n1} \overset{j}{+} \dots \overset{j}{+} x_{nn}) \overset{j}{+} a_j)) \overset{k}{+} a_k =$$

$$= \alpha_k((\alpha_j(x_{\phi(1,1)} \overset{j}{+} \dots \overset{j}{+} x_{\phi(1,n)}) \overset{j}{+} a_j) \overset{k}{+} \dots \overset{k}{+} (\alpha_j(x_{\phi(n,1)} \overset{j}{+} \dots$$

$$\dots \overset{j}{+} x_{\phi(n,n)}) \overset{j}{+} a_j) \overset{k}{+} a_k .$$

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Let e_i be the neutral element of the group $\overset{i}{+}$, $i=1,\dots,m$. If in the preceding equality we put $x_{11}=x_{\phi(1,1)}=x$, $x_{12}=x_{\phi(2,1)}=y$, $x_{in}=x_{\phi(i,n)}=\alpha_j^{-1}\overset{j}{(-a)}$, $i=1,\dots,n$; $x_{i2}=\alpha_j^{-1}e_k$, $i=2,\dots,n$; $x_{\phi(i,1)}=\alpha_j^{-1}e_k$, $i=3,\dots,n$; $x_{\phi(n,n-1)}=\alpha_j^{-1}e_k$ and $x_{i\ell}=x_{\phi(i,\ell)}=e_j$ otherwise, then we get

$$(\alpha_j x\overset{j}{+}\alpha_j y)\overset{k}{+}e_k\overset{k}{+}\dots\overset{k}{+}e_k=(\alpha_j x)\overset{k}{+}(\alpha_j y)\overset{k}{+}e_k\overset{k}{+}\dots\overset{k}{+}e_k\overset{k}{+}(e_k\overset{j}{+}e_k),$$

thus

$$\alpha_j x\overset{j}{+}\alpha_j y=(\alpha_j x)\overset{k}{+}(\alpha_j y)\overset{k}{+}(e_k\overset{j}{+}e_k).$$

So, for all $x,y\in Q$

$$x\overset{j}{+}y=x\overset{k}{+}y\overset{k}{+}f,$$

where $f=e_k\overset{j}{+}e_k$, which yields the identity

$$f_j(x_1^n)=\alpha_j(x_1\overset{k}{+}\dots\overset{k}{+}x_n)\overset{k}{+}\bar{a}_j$$

where $\bar{a}_j=nf\overset{k}{+}a_j$.

The converse part of the theorem can be proved by a straightforward computation.

The preceding theorem is a generalization of Proposition 3.1. from |6|. In fact, in |6| a WTSB $[n,m]$-groupoids are defined (as a generalization of C^{n+1}-systems introduced in |4|). An $[n,m]$-groupoid is WTSB iff it is bisymmetric and all its components operations are totally symmetric n-quasigroups. If $Q(f)$ is a WTSB $[n,m]$-groupoid, then from Theorem 4 it follows that

$$f_i(x_1^n)=\alpha_i\sum_{j=1}^{n}x_j+a_i,\quad i=1,\dots,m.$$

If $f_i(x_1^n)=y$, then putting in the last equality $x_2=\dots=x_n=0$, because of the total symmetry of f_i, we get

$$\alpha_i x_1+\alpha_i y+a_i=0,\quad\text{and}\quad \alpha_i y+a_i=x_1,$$

which implies $\alpha_i x_1=-x_1$ for all $x_1\in Q$.

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REZIME

O BISIMETRIČNIM [n,m]-GRUPOIDIMA

U ovom radu razmatrani su bisimetrični [n,m]-grupoidi. Pokazano je da je svaki bisimetričan [n,m]-grupoid Q(f) koji ima bar jednu komponentnu operaciju čiji je kodomen Q, komutativan. Posledica ovog tvrdjenja je da ne postoji prava bisimetrična [n,m]-kvazigrupa (što je dokazano u |6|). Takodje je dokazano da iz komutativnosti i uzajamne medijalnosti komponentnih operacija sledi bisimetričnost [n,m]-grupoida. Data je jena karakterizacija komponenata bisimetričnog [n,m]-grupoida koje su n-kvazigrupe. Navedeni su potrebni i dovoljni uslovi da [n,m]-grupoid bude bisimetričan sa komponentama koje su n-kvazigrupe.

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CYCLIC n-QUASIGROUPS

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ABSTRACT

In this paper cyclic n-groupoids (Definition 1), i.e. cyclic n-quasigroups (because every cyclic n-groupoid is necessarily a cyclic n-quasigroup) are considered. Different equivalent definitions of a cyclic n-groupoid are given. Examples of cyclic n-quasigroups are listed. Circular parastrophes of an n-quasigroup, which are suitable for the study of cyclic n-quasigroups, are defined. It is determined which parastrophes of a cyclic n-quasigroup are cyclic. It is shown that an n-quasigroup which is isotopic to a cyclic n-quasigroup must be isotopic to all its circular parastrophes and conditions under which its parastrophes are isotopic to a cyclic n-quasigroup are given. Some consequences which follow from the assumption that an n-quasigroup is isotopic to one of its parastrophes are obtained. Using these consequences a theorem which gives necessary and sufficient conditions for an n-quasigroup to be isotopic to a cyclic n-quasigroup is proved.

First we give some basic definitions and notations. Other notions from the theory of n-ary quasigroups can be found in [1].

The sequence $x_m, x_{m+1}, \ldots, x_n$ we denote by $\{x_i\}_{i=m}^n$ or x_m^n. If $m > n$, then x_m^n will be considered empty. The sequence $x, x, \ldots, x$ (n times) will be denoted by $\overset{n}{x}$.

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An n-groupoid (Q,A) is called an n-quasigroup iff the equation $A(a_1^{i-1}, x, a_{i+1}^{n}) = b$ has a unique solution x for every $a_1^n, b \in Q$ and every $i \in N_n = \{1,\dots,n\}$.

An n-quasigroup (Q,A) is isotopic to n-quasigroup (Q,B) iff there exists a sequence $T = (\alpha_1^{n+1})$ of permutations of Q such that the following identity

$$B(x_1^n) = \alpha_{n+1}^{-1} A(\{\alpha_i x_i\}_{i=1}^{n})$$

holds. T is called an isotopism, and by $A^T = B$ we denote that A is isotopic to B by T. T^{-1} is defined by $T^{-1} = (\{\alpha_i^{-1}\}_{i=1}^{n+1})$.

If (Q,A) is an n-quasigroup and $\sigma \in S_{n+1}$, where S_{n+1} is the symmetric group of degree n+1, then the n-quasigroup A^σ defined by

$$A^\sigma(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)} \iff A(x_1^n) = x_{n+1},$$

is called a σ-parastrophe (or simply parastrophe) of A. If $\sigma, \tau \in S_{n+1}$ then $(A^\sigma)^\tau = A^{\sigma\tau}$ and

$$A(x_{\sigma 1}^{\sigma n}) = x_{\sigma(n+1)} \iff A^\tau(x_{\sigma\tau(1)}^{\sigma\tau(n)}) = x_{\sigma\tau(n+1)}.$$

If $T = (\alpha_1^{n+1})$ is an isotopy of A, then $(A^T)^\sigma = (A^\sigma)^{T^\sigma}$, where $T^\sigma = (\alpha_{\sigma 1}^{\sigma(n+1)})$.

An n-quasigroup (Q,A) is called idempotent iff for every $x \in Q$ $A(\overset{n}{x}) = x$.

An n-quasigroup (Q,A) is called totally symmetric iff A coincides with all its parastrophes.

DEFINITION 1. *An n-groupoid (Q,A) is called cyclic iff the following identity holds*

$$A(A(x_1^n), x_1^{n-1}) = x_n.$$

A binary groupoid or quasigroup $(Q,\cdot)$ is called semi-symmetric iff it satisfies the identity $(xy)x = y$. So, cyclic n-groupoids defined in Definition 1 are a generalization of semi-symmetric groupoids. Semi-symmetric quasigroups were investigated

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in various directions. Some results on semi-symmetric quasigroups (mostly obtained by A. Sade) are used in the enumeration and classification of latin squares and some classes of semi-symmetric quasigroups are related to geometry of plane curves. There is also a connection between indempotent semi-symmetric quasigroups and balanced incomplete block designs ([3]) and some other combinatorial designs ([2]).

The definition of a cyclic n-groupoid can be also given in another form. The next definition is equivalent to Definition 1.

DEFINITION 1'. *An n-groupoid* (Q,A) *is cyclic iff for all* $x_1^{n+1} \in Q$

$$A(x_1^n) = x_{n+1} \Longrightarrow A(x_{n+1}, x_1^{n-1}) = x_n .$$

Using the preceding implication it is easy to obtain the following definition equivalent to the preceding ones.

DEFINITION 1''. *An n-groupoid* (Q,A) *is cyclic iff for every* $i \in N_n$ *and all* $x_1^{n+1} \in Q$

$$A(x_1^n) = x_{n+1} \Longleftrightarrow A(x_{i+1}^{n+1}, x_1^{i-1}) = x_i .$$

If (Q,A) is a cyclic n-groupoid, then for all $i \in N_n$ and all a_1^{i-1}, $a_{i+1}^{n+1} \in Q$, the equation

$$A(a_1^{i-1}, x, a_{i+1}^n) = a_{n+1} \tag{1}$$

is by Definition 1'' equivalent to

$$A(a_{i+1}^{n+1}, a_1^{i-1}) = x .$$

This means that the equation (1) has a unique solution for x, hence the following proposition is valid.

PROPOSITION 1. *Every cyclic n-groupoid is an n-quasigroup.*

We now give some examples of cyclic n-quasigroups.

1. Let $(Q,+)$ be an arbitrary Abelian group (binary), $n \geq 2$, a an arbitrary element from Q, and ϕ an automorphism of

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the group $(Q,+)$ such that $\phi a = -a$, and if n is even then for all $x \in Q$, $\phi^{n+1}x = -x$, and if n is odd then ϕ^{n+1} is the identity mapping. Then by

$$(2) \quad A(x_1^n) = \phi x_1 - \phi^2 x_2 + \phi^3 x_3 - \ldots + (-1)^{n+1}\phi^n x_n + a$$

a cyclic n-quasigroup (Q,A) is defined. Among cyclic n-quasigroups defined by (2) there are n-quasigroups which are neither totally symmetric nor idempotent and which are not n-groups.

An automorphism of an arbitrary Abelian group, which satisfies the given conditions is given by $\phi : x \mapsto -x$, and the cyclic n-quasigroup obtained that way for n even need not be idempotent nor n-group, but it is always totally symmetric. If n is odd, ϕ the identity mapping and $a = 0$, a cyclic n-quasigroup which is not totally symmetric can be obtained.

In an arbitrary ring with unity $(R,+,.)$ the mapping $\phi : x \mapsto bx$, where b is an invertible element, is an automorphism of the additive group $(R,+)$. If b is such that for n even $b^{n+1} = -1$, and for n odd $b^{n+1} = 1$, then by (2) a cyclic n-quasigroup (R,A) is defined.

If R is the ring Z_q of integers modulo q, then, for example, for $q = 15$, $b = 4$ and an arbitrary odd n a cyclic n-quasigroup of order 15 is obtained, for $q \in \{4,61,122\}$, $b = 3$, $n = 4$ a cyclic 4-quasigroup of order q is obtained etc.

2. Let $(Q,\cdot)$ be an arbitrary group, α an automorphism of this group such that α^{n+1} is the identity mapping, a an element of the center of the given group such that $\alpha a = a$. Then by

$$A(x_1^n) = \alpha x_1^{-1} \alpha^2 x_2^{-1} \ldots \alpha^n x_n^{-1} a$$

a cyclic n-quasigroup is defined.

Now we shall define one kind of parastrophes which are suitable for the study of cyclic n-quasigroups.

DEFINITION 2. *If (Q,A) is an n-quasigroup, $i \in N_n \cup \{0\}$ then each of the n+1 parastrophes A^{σ_i}, $i = 1, \ldots, n+1$, defined by $\sigma_i : a \mapsto a+i \pmod{(n+1)}$ is called circular parastrophe of the n-quasigroup (Q,A).*

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From the definitions 1', 1" and 2 the following propositions can be obtained.

THEOREM 1. *An n-quasigroup (Q,A) is cyclic iff A coicides with its parastrophe* A^{σ_i}, *where* σ_i *is a circular parastrophe such that* i *is relatively prime to* n+1.

THEOREM 2. *An n-quasigroup (Q,A) is cyclic iff it coincides with all its circular parastrophes.*

In the binary case ([4]) all parastrophes of a cyclic quasigroup are cyclic. But, for an n-quasigroup, $n > 2$, this is not the case. Let (Q,A) be a cyclic n-quasigroup and A^σ its parastrophe. For all $x_1^{n+1} \in Q$ and every $i \in N_n$ we have

$$A^\sigma(x_1^n) = x_{n+1} \iff A(x_{\sigma^{-1}1}^{\sigma^{-1}n}) = x_{\sigma^{-1}(n+1)} \iff A^{\sigma_i}(x_{\sigma^{-1}\sigma_i^{-1}1}^{\sigma^{-1}\sigma_i^{-1}n}) =$$

$$= x_{\sigma^{-1}\sigma_i(n+1)} \iff A(x_{\sigma^{-1}(1+i)}^{\sigma^{-1}(n+i)}) = x_{\sigma^{-1}(n+1+i)} \iff$$

$$\iff A^\sigma(x_{\sigma^{-1}(1+i)}^{\sigma^{-1}(n+i)}) = x_{\sigma^{-1}(n+1)+i}$$

where all indices are reduced modulo n+1. Hence, if σ is such that for an $i \in N_n$ and all $t \in N_n$

$$\sigma^{-1}(\sigma(t)+i) = t+1 \pmod{(n+1)},$$

i.e.

$$\sigma(t+1) = \sigma(t) + i \pmod{(n+1)}, \tag{3}$$

then the parastrophe A^σ is cyclic. So we have proved

THEOREM 3. *Let (Q,A) be a cyclic n-quasigroup and* A^σ *its parastrophe such that* σ *satisfies (3). Then* A^σ *is a cyclic n-quasigroup.*

Of course, the circular parastrophes always satisfy (3), but there are other noncircular parastrophes which satisfy (3). Other parastrophes of a cyclic n-quasigroup which do not satisfy the condition (3) need not be cyclic. If n=2 then (3) is satisfied for all permutations $\sigma \in S_3$.

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We shall now consider isotopes of cyclic n-quasigroups.

THEOREM 4. *If an* n*-quasigroup* A *is isotopic to a cyclic* n*-quasigroup* B, *then* A *is isotopic to each of its circular parastrophes* A^{σ^i}. *Then all parastrophes of* A *which satisfy* (3) *are isotopes of cyclic* n*-quasigroups.*

P r o o f. Since B is a cyclic n-quasigroup, it coincides with all its circular parastrophes. Corresponding parastrophes of isotopic n-quasigroups are isotopic, hence, the circular parastrophes of A are isotopic to the corresponding parastrophes of B, which means that all circular parastrophes of A are isotopic to B. Since the isotopism is transitive, it follows that A is isotopic to each of its circular parastrophes.

The parastrophes of the cyclic n-quasigroup B which satisfy (3) are cyclic n-quasigroups, hence, the corresponding parastrophes of A are isotopes of cyclic n-quasigroups.

THEOREM 5. *Let* (Q,A) *be an* n*-quasigroup isotopic to its parastrophe* A^σ *by an isotopism* $T = (\alpha_1^{n+1})$, $A^T = A^\sigma$, *where* $\sigma \in S_{n+1}$ *is an arbitrary cycle of length* n+1. *Then there exist a permutation* $\theta \in S_Q$ *and an* n*-quasigroup* (Q,B) *which is isotopic to* A, *such that* B *is isotopic to* B^σ *by the isotopism* $(1,\dots,1,\theta^{-1}\alpha_{n+1}\alpha_{\sigma(n+1)}\cdots\alpha_{\sigma^n(n+1)}\theta)$.

P r o o f. Let B be an arbitrary isotope of A, $B = A^S$, $S = (\beta_1^{n+1})$. Since $A^T = A^\sigma$, it follows

$$(((B^{S^{-1}})^T)^{\sigma^{-1}})^S = B.$$

For every n-quasigroup (Q,C) we have $(C^{\sigma^{-1}})^S = (C^S)^{\sigma^{-1}}$, hence,

$$((B^{S^{-1}T})^{S^\sigma})^{\sigma^{-1}} = B,$$

and

$$B^{S^{-1}TS^\sigma} = B^\sigma,$$

which means that every isotope B of the n-quasigroup A is isotopic its parastrophe B^σ.

Now we show that there exists an isotope B of A, such that there is an isotopism of B to B^σ which has the form described in the theorem. Since $S^{-1}TS^\sigma = (\{\beta_i^{-1}\alpha_i\beta_{\sigma i}\}_{i=1}^{n+1})$, then

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$B^{(\{\beta_i^{-1}\alpha_i\beta_{\sigma i}\}_{i=1}^{n+1})} = B$. If we put

$$\beta_1^{-1}\alpha_1\beta_{\sigma 1} = 1, \ldots, \beta_n^{-1}\alpha_n\beta_{\sigma n} = 1,$$

then

$$\alpha_1\beta_{\sigma 1} = \beta_1, \ldots, \alpha_n\beta_{\sigma n} = \beta_n .$$

If we take β_{n+1} to be an arbitrary permutation from Q, then from the i-th of the preceding equations, where $i=\sigma^{-1}(n+1)$, we get β_i, then from the j-th equation, where $j=\sigma^{-1}i=\sigma^{-2}(n+1)$, we get β_j and so on, at the end we get $\beta_{\sigma^{-n}(n+1)}$. Since σ is a cycle of length n+1 we have obtained all permutations $\beta_1, \ldots, \beta_n$:

$$(4)\qquad \begin{aligned} \beta_{\sigma^{-1}(n+1)} &= \alpha_{\sigma^{-1}(n+1)}\beta_{n+1}, \\ \beta_{\sigma^{-2}(n+1)} &= \alpha_{\sigma^{-2}(n+1)}\beta_{\sigma^{-1}(n+1)} = \alpha_{\sigma^{-2}(n+1)}\alpha_{\sigma^{-1}(n+1)}\beta_{n+1} \\ &\cdots\cdots\cdots\cdots\cdots\cdots\cdots\cdots \\ \beta_{\sigma^{-n}(n+1)} &= \alpha_{\sigma^{-n}(n+1)}\alpha_{\sigma^{-n+1}(n+1)}\ldots\alpha_{\sigma^{-1}(n+1)}\beta_{n+1} . \end{aligned}$$

Then the (n+1)-th component of the isotopism $S^{-1}TS^{\sigma}$ has the form:

$$\begin{aligned} \beta_{n+1}^{-1}\alpha_{n+1}\beta_{\sigma(n+1)} &= \beta_{n+1}^{-1}\alpha_{n+1}\beta_{\sigma^{-n}(n+1)} = \\ &= \beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma^{-n}(n+1)}\beta_{\sigma^{-n+1}(n+1)} = \\ &= \ldots = \beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma^{-n}(n+1)}\alpha_{\sigma^{-n+1}(n+1)}\ldots\alpha_{\sigma^{-1}(n+1)}\beta_{n+1} = \\ &= \beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma(n+1)}\alpha_{\sigma^2(n+1)}\ldots\alpha_{\sigma^n(n+1)}\beta_{n+1} . \end{aligned}$$

So, we have obtained that for an arbitrary $b_{n+1} \in S_Q$ the isotope B of A by the isotopism $S=(\beta_1^{n+1})$, where β_i, i=1,...,n are given by (4), $B=A^S$, is an n-quasigroup which is isotopic to its parastrophe B^{σ} by the isotopism $(1, \ldots, 1, \beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma(n+1)}\ldots\alpha_{\sigma^n(n+1)}\beta_{n+1})$.

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REMARK. A special case of this theorem, when n=2, is a generalization of a theorem from [3], where the corresponding proposition is proved for only one parastrophe A^{σ}, $\sigma \in S_3$.

THEOREM 6. *An n-quasigroup (Q,A) is isotopic to a cyclic n-quasigroup iff there exists a circular parastrophe* A^{σ_i}, *where* i *is relatively prime to* n+1, *such that* A *is isotopic to* A^{σ_i} *by an isotopism* $T = (\alpha_1^{n+1})$ *where* $\alpha_{n+1}\alpha_{\sigma_i(n+1)}\cdots\alpha_{\sigma_i^n(n+1)} = 1$.

P r o o f. If there is a parastrophe A^{σ_i} which satisfies the conditions given in the theorem, then by the preceding theorem A is isotopic to an n-quasigroup B which coincides with its parastrophe B^{σ_i}. If B coincides with B^{σ_i}, then by Theorem 1 B is cyclic, hence, A is isotopic to a cyclic n-quasigroup.

Conversely, let A be isotopic to a cyclic n-quasigroup B, $A^S = B$, $S = (\beta_1^{n+1})$. Let A^{σ_i} be a circular parastrophe such that i is relatively prime to n+1. Then σ_i is a cycle of lenght n+1 and $(A^{\sigma_i})^{S^{\sigma_i}} = B^{\sigma_i}$. Since B is cyclic $B = B^{\sigma_i}$. From $A^S = B$, $(A^{\sigma_i})^{S^{\sigma_i}} = B$, it follows

$$A^{S(S^{\sigma_i})^{-1}} = A^{\sigma_i},$$

so, $S(S^{\sigma_i})^{-1}$ is an isotopism of A to A^{σ_i}. If we denote this isotopism by $T = S(S^{\sigma_i})^{-1} = (\alpha_1^{n+1})$, we shall have $S^{-1}TS^{\sigma_i} = I = (1,\ldots,1)$, i.e.

$$\beta_k^{-1}\alpha_k\beta_{\sigma_i k} = 1, \quad k=1,\ldots,n+1 .$$

From these equalities, analogously as it is done in the proof of the preceding theorem, we get

$$\beta_{n+1}^{-1}\alpha_{n+1}\alpha_{\sigma_i(n+1)} \cdots \alpha_{\sigma_i^n(n+1)}\beta_{n+1} = 1,$$

i.e.

$$\alpha_{n+1}\alpha_{\sigma_i(n+1)} \cdots \alpha_{\sigma_i^n(n+1)} = 1.$$

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REZIME

CIKLIČKE n-KVAZIGRUPE

U ovom radu razmotreni su ciklički n-grupoidi (definicija 1), odnosno n-kvazigrupe (jer je svaki ciklički n-grupoid n-kvazigrupa). Date su različite ekvivalentne definicije cikličkih n-kvazigrupa. Navedeni su primeri cikličkih n-kvazigrupa, medju njima i primeri n-kvazigrupa koje nisu idempotentne, ni totalno simetrične. Definisani su kružni parastrofi koji su pogodni za izučavanje cikličkih n-kvazigrupa. Utvrdjeno je koji parastrofi cikličke n-kvazigrupe moraju biti ciklički. Pokazano je da je n-kvazigrupa koja je izotopna cikličkoj n-kvazigrupi izotopna svim svojim kružnim parastrofima i navedeni su uslovi pod kojima su njeni parastrofi izotopni cikličkim n-kvazigrupama. Iz pretpostavke da je n-kvazigrupa izotopna svom parastrofu odredjenog tipa izvedene su razne posledice. Na osnovu tih posledica dokazana je teorema koja daje potrebne i dovoljne uslove pod kojima je n-kvazigrupa izotopna cikličkoj n-kvazigrupi.

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POWER REGULAR SEMIGROUPS

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ABSTRACT

In the present paper power intra-regular and power (left, right) regular semigroups are considered and in this way the theory of R.Croisot, |7| is generalized.

R.Croisot considered in |7| the semigroups which are unions of simple semigroups (so called intra-regular semigroups) i.e. semigroups with the property that each element is in a simple subsemigroup. He also considered left regular, right regular and regular semigroups.

Let P be one of the following properties defined below: power intra-regular, power left regular, power regular, power inverse or power orthodox. In this paper we shall consider the semigroups with the property P.

For undefined notions and notations we refer to |6|, |10| and |13|.

1. POWER INTRA-REGULAR SEMIGROUPS

DEFINITION 1.1. *A semigroup S is power intra-regular if for every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \in Sa^{2m}S$.

LEMMA 1.1. *S is power intra-regular if and only if for every* $a \in S$ *there exists* $m \in N$ *such that* $a^m J a^{2m}$ (1).

(1) $aJb \Leftrightarrow J(a) = J(b)$

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THEOREM 1.1. *S is a power intra-regular semigroup if and only if some power of each element of S lies in a simple subsemigroup.*

P r o o f. Let S be a power intra-regular semigroup. Then for an arbitrary $a \in S$ there exists $m \in N$ such that $J(a^m) = Sa^mS$ and it is clear that

(1) $$Sa^mS = Sa^{2m}S, \quad \text{(Lemma 1.1.)}$$

Assume that $b,c \in J_{a^m}$(2). Then cJa^m and c^2Ja^{2m}, so by (1) we have that cJc^2. Similarly, bJb^2. From this it follows that

(2) $$J(c) = Sc^kS, \quad J(b) = Sb^rS$$

for some $k,r \in N$. Since

$$Sc^kSSc^kS \subseteq Sc^kS$$

$$Sc^kS = Ss_1(c^k)^{2p}s_2S, \text{ for some } p \in N \text{ and } s_1,s_2 \in S$$

$$= Ss_1s_3(c^{kp})^{2h}s_4c^{kp}s_2S \subseteq Sc^kSSc^kS$$

$$= Ss_1s_3(c^{kp})^{2h}s_4c^{kp}s_2S \subseteq Sc^kSSc^kS$$

it follows that

(3) $$J(c) = J^2(c) .$$

Similarly

(4) $$J(b) = J^2(b) .$$

From (3) and (4) we have

(5) $$J(c) = J(c)J(c) = J(c)J(b) = Sc^kSSb^rS ,$$

(since cJb).

From $S(c^kSSb^r)^2S \subseteq Sc^kSSb^rS = S(S(c^kSSb^r)^{2h}S)S \subseteq S(c^kSSb^r)^2S$

we have

(6) $$S(c^kSSb^r)^2S = Sc^kSSb^rS .$$

By (5) and (6) it follows that

(7) $$Sc^kSSb^rS = S(c^kSSb^r)(c^kSSb^r)S \subseteq Sb^rc^kS$$

(2) J_{a^m} is a J-class of element a^m.

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(8) $Sb^rc^kS \subseteq S(b^rc^k)^{2t}S \subseteq \ldots \subseteq Sc^kSSb^rS$.

Hence, by (7) and (8) we have

(9) $Sc^kSSb^rS = Sb^rc_kS$.

Now by (5) and (9) we have $J(c) = J(c)J(b) = Sb^rc^kS \subseteq SbcS \subseteq J(bc)$ and since $J(bc) \subseteq J(b)J(c)$ we have $J(bc) = J(c)$. Hence, J_{a^m} is a subsemigroup of S. We shall show that J_{a^m} is simple. If $b,d \in J_{a^m}$, then $b, bd^3b \in J_{a^m}$ and

$$b = xbd^3by .$$

From this it follows that $b = x^2bd^3bybd^3y$. Put $u = xbd$. Then $b = xud^2by$. Hence bJu. Similarly we have that $db\,y = vJb$. Therefore, $b = (xbd)d(dby)$, $(xbd, dby \in J_{a^m})$, i.e. J_{a^m} is a simple subsemigroup of S.

Conversely, if for an arbitrary $a \in S$ there exists $m \in N$ such that a^m is in a simple subsemigroup P of S, then $a^m \in Pa^{2m}P \subseteq Sa^{2m}S$.

A semigroup S is intra-regular if $a \in Sa^2S$ for every $a \in S$ |6|.

COROLLARY 1.1. *(|6|, |7|). S is intra-regular if and only if S is a union of the simple subsemigroups of S .*

THEOREM 1.2. *Every principal left ideal of S is a simple subsemigroup of S if and only if* $a \in SabSa$ *for every* $a,b \in S$.

P r o o f. If every principal left ideal of S is simple and L(a) is an arbitrary principal left ideal of S, then $L(a) = L(a)\,x\,L(a)$ for every $x \in L(a)$. For $b \in S$ we have that $ba \in L(a)$ and $L(a) = L(a)baL(a) \subseteq L(a)bL(a) \subseteq L(a)$, i.e. $L(a) = L(a)bL(a)$ for every $b \in S$. Hence, $a = aba$ or $a \in Saba$ or $a \in abSa$ or $a \in SabSa$ for every $b \in S$. From this if follows that $a \in SabSa$ for every $a,b \in S$.

Conversely, if $x,y \in L(a)$, then

$x = a$, $y = a \Longrightarrow x = \alpha aa\beta a \in L(a)yL(a)$

$x = za$, $y = a \Longrightarrow x = \alpha(za)a\beta(za) \in L(a)yL(a)$

$x = a$, $y = ua \Longrightarrow x = \alpha a(ua)\beta a \in L(a)yL(a)$

$x = za$, $y = ua \Longrightarrow x = \alpha(za)(ua)\beta(za) \in L(a)yL(a)$.

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Hence, L(a) is a simple subsemigroup of S.

PROBLEM 1.1. Do the following formulas

$(\forall a \in S)(\exists m \in N)(a^m \in Sa^{m+1}S)$, $(\forall a \in S)(\exists m \in N)(a^m \in Sa^{2m}S)$

define the same class of semigroups?

PROBLEM 1.2. Describe the class of semigroups with the property that each proper left ideal is a power intra-regular (intra-regular, simple) semigroup.

REMARK. Semigroups in which every proper left ideal is a completely simple semigroup are described in [4].

2. POWER LEFT REGULAR SEMIGROUPS

DEFINITION 2.1. *A semigroup* S *is power left regular if for every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \in Sa^{m+1}$.

Analogously we define a power right regular semigroup.

THEOREM 2.1. *The following conditions are equivalent on a semigroup* S :

(i) S *is power left regular*

(ii) *For every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \mathcal{L} a^{m+1}$

(iii) *Some power of each element of* S *lies in a left simple subsemigroup of* S ;

(iv) *Every principal left ideal of* S *is power left regular.*

P r o o f. (i) $\Leftrightarrow$ (ii) $\Leftrightarrow$ (iv) and (iii) $\Rightarrow$ (i) follow immediately. (i) $\Rightarrow$ (iii). Let S be a power left regular semigroup. Then by (i) and (ii) we have that for $a \in S$ there exists $m \in N$ such that $a^m \mathcal{L} a^{2m}$. If $b,c \in L_{a^m}$ then $b \mathcal{L} bc \mathcal{L} c^2$ (since $\mathcal{L}$ is a right congruence) and from $c \mathcal{L} a^m$ it follows that $c^2 \mathcal{L} a^{2m}$, so $c^2 \mathcal{L} a^m$. Hence $c^2 \mathcal{L} c$. Therefore, $bc \mathcal{L} c$, i.e. L_{a^m} is a subsemigroup of S. If $b,d \in L_{a^m}$, then $b, bd \in L_d$, so $b = xbd$ for some $x \in S^1$. From

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this we have $b = x^k bd^k$ for every $k \in N$. Let us put $c = xb$. It is sufficient to show that $c \in L_b$. Since S is power left regular we have that $x^k = yx^{k+1}$ for some $y \in S$ and $b = xbd = x^k bd^k = yx^{k+1}bd = yxx^k bd = yxb = yc$.

Hence, $b = yc$, $c = xb$, i.e. $c \in L_b$. Therefore, $L_{a^m} = L_b$ is a left simple subsemigroup of S.

Theorem 2.1. is a generalization of Theorem 4.2. |6|.

COROLLARY 2.1. *Every proper subsemigroup of S is power left regular if and only if S is periodic.*

LEMMA 2.1. |5| *Every proper left ideal of L is minimal (left simple) if and only if L contains exactly one minimal left ideal or S sontains exactly two minimal left ideals* L_1 *and* L_2 *and* $S = L_1 \cup L_2$.

The following lemma is known.

LEMMA 2.2. *The union of all minimal left ideals of S is a(twosided) ideal of S , and is a kernel of S .*

LEMMA 2.3. |5|. *Let I be a left (twosided) ideal of S. If K is a left simple (simple) subsemigroup of S and* $K \cap I \neq \emptyset$, *then* $K \subseteq I$.

LEMMA 2.4. |5|. *Let I be a proper twosided ideal of S which is not contained as a proper subset in a left ideal* $L \neq S$. *Then* $S \setminus I$ *is a left simple semigroup or* $S \setminus I = \{a\}, a^2 \in I$.

THEOREM 2.2. *Every proper left ideal of a semigroup S is minimal if and only if one of the following conditions holds:*

1^o *S has a kernel K which is a left simple subsemigroup of S and* $S \setminus K$ *is a left simple subsemigroup of S ;*

2^o *S has a kernel K which is a left simple semigroup and* $S \setminus K = \{a\}, a^2 \in K$;

3^o *S contains exactly two minimal left ideals* L_1 *and* L_2 *and* $S = L_1 \cup L_2$.

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P r o o f. If all proper left ideals of S are minimal, then by Lemma 2.1. we have two cases. Assume that S has exactly one minimal left ideal K. Then by Lemma 2.2. K is a twosided ideal of S and it is the kernel of S. By Lemma 2.4. we have that $S \setminus K$ is a left simple subsemigroup of S or $S \setminus K = \{a\}$, $a^2 \in K$. If S contains exactly two minimal left ideals, then by Lemma 2.1. we have the case 3°.

Conversely, suppose that 1° holds. Let K be a kernel of S and let K be left simple. If L is a proper left ideal of S, then $K \cap L \neq \emptyset$, so $K \subseteq L$. If $K \neq L$, then $L \cap (S \setminus K) \neq \emptyset$ and by Lemma 2.3. we have that $S \setminus K \subseteq L$. Hence, $S = K \cup (S \setminus K) \subseteq L$, which is not possible. Therefore, $K = L$, i.e. K is the unique proper left ideal od S. The case 2°. If L is a proper left ideal of S and $L \neq K$, then from $K \subseteq L$ it follows that $L = K \cup \{a\} \neq S$ which is not possible.

A subsemigroup B of S is a bi-ideal of S if $BSB \subseteq B$, |6|.

COROLLARY 2.2. *Every proper bi-ideal of S is minimal if and only if one the following condition holds:*

1° S *has a kernel* G *which is a group and* $S \setminus G$ *is a group;*

2^0 S *has a kernel* G *which is a group and* $S \setminus G = \{a\}, a^2 \in G$;

3° S *is a left group* $I \times G$ ($|I| = 2$) *or* S *is a right group* $G \times J$ ($|J| = 2$).

REMARK. The semigroups described in Corollary 2.2 are, in fact, the F-semigroups considered by Schwarz, |17|.

A semigroup S is a band Y of left (right) ideals L_i ($i \in Y$) if

$$S = \bigcup_{i \in Y} L_i, \quad L_i \cap L_j = \emptyset, \ (i \neq j), \ |3| .$$

PROPOSITION 2.1. |3|. S *is a left (right) zero band of semigroups from the class* K *if and only if* S *is a band of right (left) ideals from* K.

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THEOREM 2.3. *The following conditions are equivalent on a semigroup* S :

(i) *Every principal left ideal of* S *is a left simple subsemigroup of* S ;

(ii) S *is a right zero band of left simple semigroups;*

(iii) $a \in Sba$ *for every* $a,b \in S$.

P r o o f. (i) => (ii). If all principal left ideals of S are left simple, then the principal left ideals are minimal, so the principal left ideals are disjoint. From this and Proposition 2.1. it follows that S is a right zero band of left simple semigroups.

(ii) => (iii). If S is a right zero band Y of left simple semigroups S_α ($\alpha \in Y$), then for $a \in S_\alpha$, $b \in S_\beta$ we have

$$ba \in S_\beta S_\alpha \subseteq S_{\beta\alpha} \subseteq S_\alpha \ , \quad \text{so} \quad a \in S_\alpha ba \subseteq Sba \ .$$

(iii) => (i). Let condition (iii) hold. Assume $a \in S$ and $x,y \in L(a)$. Then we have:

(a) $x = a$, $y = a$. Then $x = a \in Saa \subseteq L(a)y$. Hence,

(*) $L(a) = L(a)y$ for every $y \in L(a)$

(b) $x = za$, $y = a$. Then $x = za \in zSaa \subseteq L(a)y$, i.e. condition (*) holds.

(c) $x = a$, $y = ua$. Then $x = a \in S(au)a \subseteq L(a)ua \subseteq L(a)y$ i.e. (*) holds.

(d) $x = za$, $y = ua$. Then $x = za \in zS(au)a \subseteq L(a)ua = L(a)y$ i.e. (*) holds. By (a),(b),(c) and (d) we have that L(a) is left simple.

3. POWER REGULAR SEMIGROUPS

DEFINITION 3.1. S *is power regular if for every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \in a^m S a^m$.

PROPOSITION 3.1. *An element* $a \in S$ *is power regular if and only if there exists* $m \in N$ *and an idempotent* $e \in S$ *such that*

(1)' $a^m S^1 = eS$.

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P r o o f. If $a \in S$ is power regular, then $a^m = a^m x a^m$ for some $m \in N$ and $x \in S$ and $e = a^m x$ is an idempotent such that $ea^m = a^m$. Therefore, (1)' holds.

Conversely, if (1) holds, then $a^m = ex$ for some $x \in S^1$, so $ea^m = e^2 x = ex = a^m$ and $e = a^m y$ for some $y \in S$. It follows from this that $a^m = ea^m = a^m y a^m$. Hence, a is power regular.

A semigroup S is <u>regular</u> if $a \in aSa$ for every $a \in S$ |6|

COROLLARY 3.1. |6|. *An element* $a \in S$ *is regular if and only if there exists an idempotent* $e \in S$ *such that* $aS^1 = eS$.

DEFINITION 3.2. S *is power completely regular if for every* $a \in S$ *there exists* $m \in N$ *and* $x \in S$ *such that* $a^m = a^m x a^m$, $a^m x = x a^m$.

PROPOSITION 3.2. *The following conditions are equivalent on a semigroup* S :

(i) S *is power completely regular ;*

(ii) *For every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \in a^m S a^{m+1}$; *(Equivalently* $a^m \in a^{m+1} S a^m$ *)*

(iii) *Every left ideal of* S *is power regular.*

P r o o f. (i) => (ii). This implication follows immediately. (ii) => (i). If for every $a \in S$ there exists $m \in N$ and $x \in S$ such that $a^m = a^m x a^{m+1}$, then $a^m \in Sa^{m+1}$, so $a^m \in Sa^{2m}$. This and Theorem 4.3. |6| imply that a^m lies in a subgroup of S. Therefore, S is a power completely regular semigroup.
(ii) => (iii). This implication follows immediately.
(iii) => (ii). For any $a \in S$ we have that $a^m \in a^m L(a^m) a^m = a^{3m} \cup a^m S a^{2m}$. Hence, condition (ii) holds.

Following Drazin |8| we say that an element x of a semigroup S is <u>pseudo-invertible</u> in S iff there is an element $\bar{x} \in S$ such that

(i) $\quad x\bar{x} = \bar{x}x$

(ii) $\quad x^n = x^{n+1}\bar{x}$ for some $n \in N$

(iii) $\quad \bar{x} = \bar{x}^2 x$.

Following Munn |12| an element $a \in S$ is said to be pseudo-invertible iff some power of a lies in a subgroup of S.

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By Proposition 3.2., Theorem 2.1. and its dual and by the results of Munn we have the following:

THEOREM 3.1. *The following conditions are equivalent on a semigroup S :*

(i) S *is power completely regular;*

(ii) *For every* $a \in S$ *there exists* $m \in N$ *such that* $a^m \in a^m S a^{m+1}$;

(iii) *Every left ideal of* S *is power regular;*

(iv) *Every element of* S *is pseudo-invertible;*

(v) *Some power of each element of* S *lies in a subgroup of* S.

REMARK. Power completely regular semigroups are treated also in |11,14,16| .

COROLLARY. 3.2. |1|. *Every proper subsemigroup of* S *is regular if and only if* S *is a monogenic semigroup of the index 2 or* S *is a union of periodic groups.*

COROLLARY 3.3. *The following conditions are equivalent on a semigroup* S :

(i) S *is a union of groups;*

(ii) *Every left ideal of* S *is regular;*

(iii) *Every principal left ideal of* S *is regular.*

4. POWER INVERSE SEMIGROUPS

A semigroup S in inverse if for every $a \in S$ there exists a unique $x \in S$ such that $a = axa$, $x = xax$, |6|.

DEFINITION 4.1. |9|. *A semigroup* S *is power inverse if for every* $a \in S$ *there exists* $m \in N$ *and a unique* $x \in S$ *such that* $a^m = a^m x a^m$ *and* $x = x a^m x$.

THEOREM 4.1. *The following conditions are equivalent on a semigroup* S :

(i) S *is power regular and for every two idempotents* e *and* f *from* S *there exists* $n \in N$ *such that* $(ef)^n = (fe)^n$;

(ii) *For every* $a \in S$ *there exists* $m \in N$ *such that* $S^1 a^m$ *and* $a^m S^1$ *contain a unique idempotent generator;*

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(iii) S *is power inverse.*

P r o o f. (i) => (ii). By Proposition 3.1. there exists an idempotent e such that $a^m S^1 = eS$. If there exists an idempotent $f \in S$ such that $a^m S^1 = fS$, then $eS = fS$, so $efe = f$ and $fe = e$. Since $(ef)^n = (fe)^n$, for some $n \in N$ we have that $e = f$.

(ii) = (iii). By Proposition 3.1. S is power regular. We have to prove the uniquenessof the inverse element of a^m. Let b and c be the inverses of a^m. Then $a^m bS = a^m S = a^m cS$, $Sba^m = Sa^m = Sca^m$ and since the idempotent is unique we have $a^m b = a^m c$, $ba^m = ca^m$, so $b = ba^m b = ba^m c = ca^m c = c$.

(iii) = (i). This implication follows by Theorem 4.6. |9|.

DEFINITION 4.2. S *is a strongly power inverse semigroup if* S *is power regular and the idempotent elements commute.*

THEOREM 4.2. S *is a strongly power inverse semigroup if and only if* S *is power inverse and the product of any two idempotents of* S *is an idempotent.*

P r o o f. By Theorem 4.1.

5. POWER ORTHODOX SEMIGROUPS

DEFINITION 5.1. *A semigroup* S *is power orthodox if* S *is a power regular in which idempotents form a subsemigroup.*

The following theorem analogous to the theorem of Reilly and Scheiblich, |15| (see also |10|).

THEOREM 5.1. *If* S *is a power regular semigroup, then the following statements are equivalent:*

(i) S *is power orthodox:*

(ii) *If for any* $a,b \in S$ *there exist* $m,n \in N$ *and* $x,y \in S$ *such that* $a^m = a^m x a^m$, $x = xa^m x$, $b^n = b^n y b^n$, $y = yb^n y$ *then* yx *is an inverse of* $a^m b^n$.

P r o o f. (i) => (ii). Similarly as (A) => (B) in Theorem IV 1.1. |10|. (ii) => (i). If e and f are idempotents

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of S, then each is an inverse of itself and by (ii) we have that $ef = effeef = (ef)^2$.

COROLLARY 5.1. *If S is power orthodox, then every inverse of an idempotent is an idempotent.*

P r o o f. Let e be an idempotent and x its inverse, i.e. $e = exe$, $x = xex$. Then xe and ex are idempotents and so each is an inverse itself. By Theorem 5.1. we have that xex is an inverse of ex^2e. Hence, $x = x(ex^2e)x = xex\ xex = x^2$.

PROBLEM 5.1. Describe the power regular semigroups in which the inverse element of each idempotents is also idempotent.

PROBLEM 5.2. A more general approach than that of power orthodox semigroups is as follows: Let S have nonempty set R of regular elements. S is called an R-semigroup if R is a (regular) subsemigroup of S.

Examples are provided by: (1) A semigroup in which the identity $axabyb = abzab$ holds is a power regular R-semigroup; (2) A commutative power regular semigroup is an R-semigroup. The power regular R-semigroups remain to be described in the general case.

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REZIME

STEPENO REGULARNE POLUGRUPE

U ovom radu razmatraju se polugrupe sa svojstvom P: stepeno intra-regularna, stepeno levo (desno) regularna, stepeno regularna, stepeno inverzna, stepeno ortodoksna.

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SEMIGROUPS WITH COMPLETELY SIMPLE KERNEL

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ABSTRACT

In the present paper some new characterizations of semigroups with a completely simple kernel by bi-ideals and left ideals are given. Also, in this paper we consider semigroups in which every proper left ideal (bi-ideal) is a completely simple semigroup and the isomorphism theorem of this semigroup is given.

INTRODUCTION

Semigroups which contain minimal ideals are considered by A.H. Clifford, |6|. If a semigroup contains at least one minimal left and at least one minimal right ideal, then it contains a completely simple kernel, or equivalently, it contains a quasi - ideal which is a group (see Theorem 3.3. |6| and Theorem 5.14. |18|). The structural theorem for this class of semigroups is is given by A.H.Clifford, |7| (see also Theorem 2.1. |13| and Theorem 1.1. |16|). The description of this class of semigroups by bi-ideals and AB-ideals is given by S.Bogdanović, |1|.

Semigroups in which every proper subsemigroup is a group are considered by G.Pollák and L.Rédei, |15|. Š.Schwarz, |17| worked on semigroups in which every proper left ideal is a group. The same class of semigroups was studied by R.Hrmová, |11| Ġ.Čupona, |8| considered first a semigroup S in which every subset $Sx \neq S$ is a group, later, semigroups in which some left ideal is a group, |9|. This result is a special case of the result given by A.Clifford, |7|. Clearly, this class of semigroups is

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is more extensive that the one considered by Schwarz and Hrmová. The result of Ď.Čupona was an impulse for S.Milić and V.Pavlović, |13|, to give a structural description of semigroups in which some ideal is a completely simple semigroup, i.e. a semigroup with completely simple kernel. It is given an extension of a Rees matrix semigroup over a group in their paper. (See, also, |7| by A.H.Clifford). In |16| P.Protić and S.Bogdanović give the description of an extension of Rees matrix semigroup over a monoid.

Here will be given some new characterizations of semigropus with completely simple kernel by bi-ideals and left ideals (Theorem 2.1.). Also, in this paper we shall consider semigroups in which every proper left ideal (bi-ideal) is a completely simple semigroup (Theorem 2.1. and Theorem 6.1.), and next, semigroups in which every proper left ideal is a right group. In section 8. some isomorphism theorems will be given. Semigropus in which every left ideal is a left group are described by Bogdanović, |2|. Various generalizations of the results of Pollák-Rédei, |15| are given in |3| and |4|. Among other results, the description of semigroups in which every proper subsemigroup is simple is given in |4| (Theorem 2.1.).

2. SEMIGROUPS WITH COMPLETELY SIMPLE KERNEL

Here will be given some new characterizations of semigroups with completely simple kernel. Let us denote $M(G;I,J;P)$ a Rees matrix semigroup over a group G.

THEOREM 2.1. *The following conditions on a semigroup* S *are equivalent:*

1° *Some left ideal of* S *is a completely simple semigroup;*
2° *Some bi-ideal of* S *is a completely simple semigroup;*
3° *Some left ideal of* S *is a left group;*
4° S *has a completely simple kernel.*

P r o o f. 1° => 2°. Let L be a left ideal of S which is a completely simple subsemigroup of S. Then $L = M(G;I,J;P)$, so the group $H_{ij} = \{(g;i,j): g \in G\}$, $(i \in I, j \in J)$

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is a bi-ideal of L. It follows that

$$H_{ij}SH_{ij} = H_{ij}SH_{ij}H_{ij} \subseteq H_{ij}SLH_{ij} \subseteq H_{ij}LH_{ij} \subseteq H_{ij} .$$

Therefore, H_{ij} is a bi-ideal of S.

$2^{o} \Rightarrow 4^{o}$ Let a bi-ideal B of S be a completely simple semigroup. Then a maximal subgroup G of B is a bi-ideal of B, so it is a bi-ideal of S. Consequently, by Theorem 1. |1|, S has a completely simple kernel.

$3^{o} \Rightarrow 4^{o}$. If a left ideal L of S is a left group, then L contains a right ideal G which is a group; G is a bi-ideal of S, so by Theorem 1. |1|, S has a completely simple kernel.

$4^{o} \Rightarrow 3^{o}$. If S contains a completely simple kernel K, then we can put $K = M(G;I,J;P)$, thus $L_j = \{(g;i,j) : g \in G\}$ $i \in I$, $j \in J$ is a left ideal of K. Semigroup L_j is a left group, so $L_j = L_j^2$, $j \in J$. It follows that $SL_j = SL_j^2 \subseteq SKL_j \subseteq KL_j \subseteq L_j$. Therefore, S has a left ideal which is a left group.

$4^{o} \Rightarrow 1^{o}$. It follows immediately.

Also, let us consider the following conditions:

5^{o} Some bi-ideal of S is a group;

6^{o} Some quasi-ideal of S is a group;

7^{o} S has at least one minimal left and at least one minimal right ideal.

According to Theorem 1. |1|, Theorem 3.2. |6| and Theorem 2.1. we get the following result.

THEOREM 2.2. *Conditions* 1^{o}-7^{o} *are equivalent on a semigroup* S .

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3. SEMIGROUPS IN WHICH EVERY PROPER LEFT IDEAL A COMPLETELY SIMPLE SEMIGROUP

LEMMA 3.1. |2|. *Let* I *be a (left) two sided ideal of a semigroup* S, *and let* K *be a (left) simple subsemigroup of* S *such that* $K \cap I \neq \emptyset$, *then* $K \subseteq I$.

LWMMA 3.2. |2| *Let* I *be a proper two-sided ideal of* S *which is not a proper subset of any proper left ideal of* S. *Then*

a) $S \setminus I$ *is a left simple semigroup*

or

b) $S \setminus I = \{a\}$, $a^2 \in I$.

LEMMA 3.3. *A left ideal* L *of a completely simple semigroup* S *is a completely simple subsemigroup of* S.

P r o o f. It is known that S is the union of groups, so for each $a \in L$ there exists $a^{-1} \in S$ such that $a = aa^{-1}a$, $aa^{-1} = a^{-1}a$, $a^{-1} = a^{-1}aa^{-1} = a^{-1}a^{-1}a \in SL \subseteq L$ Thus L is the union of groups and because of the fact that each idempotent of S is a primitive idempotent, we have that L is a completely simple semigroup, (|10|, IV Theorem 2.4.).

LEMMA 3.4. *Every proper left ideal of a semigroup* S *is a completely simple semigroup if and only if* S *has a kernel* K *and one of the following condition hold:*

1° $S = K \cong M(G;I,J;P)$;

2° $K \cong M(G;I,J;P)$, $S \setminus K$ *is a left simple subsemigroup of* S *and for each* $a \in S \setminus K$, $Ka \supseteq K$;

3° K *is a left group,* $S \setminus K = \{a\}$ *and* $a^2 \in K$.

P r o o f. Let every proper left ideal of S be a completely simple semigroup. Then by Theorem 2.1. S has a kernel K which is a completely simple semigroup.

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Assume that $K \neq S$. If L is a proper left ideal of S, then by hypothesis it is a completely simple semigroup. Obviously $K \cap L \neq \emptyset$ and by Lemma 3.1. $L \subseteq K$. Thus, K is the unique maximal left ideal of S. According to Lemma 3.2. $S \setminus K$ is a left simple subsemigroup of S or $S \setminus K = \{a\}$, $a^2 \in K$.

The case: $S \setminus K = T$ is a left simple semigroup. Let us take $a \in T$, then the left ideal $[a]_L$ generated by a is

$$[a]_L = a \cup Sa = a \cup (K \cup T)a = a \cup Ka \cup Ta = T \cup Ka$$

(because T is a left simple semigroup). If $[a]_L$ is a proper left ideal of S, then it is a completely simple semigroup and since $K \cap [a]_L \neq \emptyset$, by lemma 3.1. we have that $[a]_L \subset K$, i.e. $T \subset K$, which is impossible. Therefore $[a]_L = S$. It follows from this that $T \cup Ka = T \cup K$ and since $K \cap T = \emptyset$ we have that $K \subseteq Ka$ for each $a \in T$.

The case: $S \setminus K = \{a\}$, $a^2 \in K$. Let us assume that K is not a left group. Then there exists a proper left ideal L_j of K which is a left group and $a^2 \in L_j$. The left ideal generated by a is

$$[a]_L = a \cup Sa = a \cup (K \cup \{a\})a = a \cup a^2 \cup Ka .$$

Then $[a]_L^2 = a^2 \cup a^3 \cup a^4 \cup Ka$, thus from $[a]_L = [a]_L^2$ it follows that $a \in [a]_L^2 \subset K$, which is impossible. Therefore $[a]_L^2 \neq [a]_L$, so $[a]_L$ is not a completely simple semigroup, and $[a]_L = S$, i.e. $a \cup a^2 \cup Ka = K \cup a$. Immediately it follows that $a^2 \cup Ka \supseteq K$, thus $a^3 \cup Ka^2 \supseteq Ka$, and $K \subseteq a^2 \cup a^3 \cup Ka^2 \subset L_j$, i.e. $K \subseteq L_j$. This means that K is a left group. Contradiction. So we have proved that $S = K \cup \{a\}$, $a^2 \in K$ and K is a left group.

Conversely, if 1° holds then the proposition follows by Lemma 3.3. If 3° holds, let us take the proper left ideal L of S. Then $K \cap L \neq \emptyset$, thus $K \subseteq L$. If $K \neq L$, then $a \in L$, i.e. $S = L$, but this is impossible. Hence, $K = L$. The case 2°. If L is a proper left ideal of S and $L \subseteq K$, then the assertion follows by Lemma 3.3. If $L \not\subset K$, then $L \cap T \neq \emptyset$, so by Lemma 3.1. we have that $T \subset L$. For $a \in T$ is $K \subseteq Ka \subset KL \subset L$. Hence, $S = K \cup T = L$, i.e. $S = L$, which is not possible.

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REMARK. Lemma 3.4. can be given in the following way: Every proper left ideal of a semigroup S is a completely simple semigroup if and only if S has a kernel $K \cong M(G;I,J;P)$ and $S \setminus K = \emptyset$ or K is the unique maximal left ideal of S. (This can be seen easy from the proof of Lemma 3.4.).

EXAMPLE. The semigroup S given by

	1	2	3	4	a
1	1	2	1	2	1
2	1	2	1	2	1
3	3	4	3	4	3
4	3	4	3	4	3
a	1	2	1	2	1

has a kernel $K = \{1,2,3,4\}$ which is a rectangular band, $S \setminus K = \{a\}$, $a^2 = 1 \in K$. But S has another maximal left ideal. It is $\{1,3,a\}$ and it is not a completely simple semigroup.

Let $K = M(G;I,J;P)$ be a Rees matrix semigroup and let T be a left simple semigroup such that $K \cap T = \emptyset$.

Let $\xi : p \to \xi_p$ be a mapping of T into the semigroup $T(I)$ of all mappings of I into itself, and $\eta : p \to \eta_p$ the mapping of T into the semigroup of all sirjections of J onto J, and for $p,q \in T$ let

(i) $$\xi_{pq} = \xi_q \xi_p \,, \quad \eta_{pq} = \eta_p \eta_q \,.$$

Let

$$\phi : T \times I \to G \quad \text{and} \quad \psi : T \times J \to G$$

be a mappings with

(ii) $$\phi(pq,i) = \phi(p,i\xi_q)\phi(q,i)$$

(iii) $$\psi(pq,j) = \psi(p,j)\psi(q,j\eta_p)$$

(iv) $$p_{ji\xi_p}\phi(p,i) = \psi(p,j)p_{j\eta_p i} \,.$$

Let us define a multiplication on $\Sigma = G \times I \times J \cup T$ with

(1) $$(a;i,j)(b;k,\ell) = (ap_{jk}b;i,\ell)$$

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(2) $(a;i,j)p = (a\psi(p,j);i,j\eta_p)$

(3) $p(a;i,j) = (\phi(p,i)a;i\xi_p,j)$

(4) $pq = r \in T \Rightarrow pq = r \in \Sigma$

for all $i,k \in I$; $j,\ell \in J$; $p,q \in T$ and $a,b \in G$.

Then, Σ with a miltiplication defined above is a semigroup. We shall denote it by $M_1(G;I,J;P;T,\phi,\psi,\xi,\eta)$.

LEMMA 3.5. *A semigroup S has a kernel K which is a completely simple semigroup, $S \setminus K$ is a left simple semigroup and for each $p \in S \setminus K$, $K \supseteq Kp$ if and only if S is isomorphic with some $M_1(G;I,J;P;T,\phi,\psi,\xi,\eta)$.*

P r o o f. Let K be a completely simple kernel of S, $K = M(G;I,J;P)$, let $S \setminus K = T$ be a left simple semigroup and $K \subseteq Kp$ for each $p \in T$. Then by Theorem 1.1. |16| (see also Theorem 1.1. |13|) there exist functions ϕ,ψ,ξ and η with properties (i)-(iii) and a multiplication on S can be defined by (1)-(4). It remains to prove that the mapping η_p is a sirjection for each $p \in T$. For an arbitrary $j \in J$ we have that $(a;i,j) \in K$, i.e. there exists $(b;k,\ell) \in K$ such that

$$(a;i,j) = (b;k,\ell)p = (b\psi(p,\ell);k,\ell\eta_p) .$$

So $j = \ell\eta_p$ for some $\ell \in J$. Thus, η_p is a sirjection of J onto J.

Conversely, let η_p $(p \in T)$ be a mapping of J onto itself. Then for $(a;i,j) \in K$, $j = \ell\eta_p$ for some $\ell \in J$, so

$$(a;i,j) = (a[\psi(p,\ell)]^{-1};i,\ell)p \in Kp.$$

Thus, $K \subseteq Kp$ for each $p \in T$. The other conditions follow from the construction.

Let $K = I \times G$ be a left group, b a fixed element of G, $a \notin K$, $\xi : a \to \xi_a$, $\xi_a \in T(I)$ and $\xi_a\xi_a = \text{const}$. Let us define a multiplication on $\Sigma = I \times G \cup \{a\}$ by:

$(i,x)(j,y) = (i,xy)$ $\qquad$ $a(i,x) = (i\xi_a,bx)$

$(i,x)a = (i,xb)$ $\qquad$ $aa = (i\xi_a\xi_a,b^2)$

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for each $i,j \in I$; $x,y \in G$. Then, Σ with a miltiplication defined in this way is a semigroup, to be denoted by $M_2(G;I;a,b,\xi_a)$.

LEMMA 3.6. |2| . *A semigroup* S *contains a kernel* K *which is a left group,* $S \setminus K = \{a\}$ *and* $a^2 \in K$ *if and only if* S *is isomorphic with some* $M_2(G;I;a,b,\xi_a)$.

By Lemmas 3.4, 3.5 and 3.6. we have

THEOREM 3.1. *Every proper left ideal of a semigroup* S *is a completely simple semigroup if and only if one of the following conditions hold:*

1° $S \cong M(G;I,J;P)$

2° $S \cong M_1(G;I,J;P;T,\phi,\psi,\xi,\eta)$

3° $S \cong M_2(G;I;a,b,\xi_a)$.

4. SEMIGROUPS IN WHICH SOME LEFT IDEAL IS A GROUP

THEOREM 4.1. *The following conditions on a semigroup* S *are equivalent:*

(i) *Some left ideal of* S *is a right group;*

(ii) *Some left ideal of* S *is a group;*

(iii) S *has a kernel which is a right group.*

P r o o f. (i) => (ii). Let a left ideal L of the semigroup S be a right group. Then L contains a left ideal G which is a group, so $SG = SGG \subset S\,LG \subset L\,G \subset G$. Thus G is a left ideal of S.

(ii) => (iii). See Corollary 2. |1|.

(iii) => (i) Follows immediately.

REMARK. Semigroups in which every proper left ideal is a group are described by Š.Schwarz, |17|. Semigroups in which some left ideal is a group are described by Ǵ. Čupona, |9|.

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5. SEMIGROUPS IN WHICH EVERY PROPER LEFT IDEAL IS A RIGHT RIGHT GROUP

LEMMA 5.1. *Every proper left ideal of a semigroup* S *is a right group if and only if one the following conditions hold:*

1° S *has a kernel* K *which is a right group and* $S \cap K = \emptyset$ *or* $S \setminus K = T$ *is a left simple semigroup and* $K \subseteq Ka$ *for each* $a \in T$;

2° S *has a kernel* K *which is a group and* $S \setminus K = \{a\}$, $a^2 \in K$.

P r o o f. If every proper left ideal of S is a right group, then by Lemma 3.4. there are three cases. Let the kernel K of S be a left group, $S \setminus K = \{a\}$, $a^2 \in K$. K has to be a right group also. Thus K is a group. The other cases and the converse follow immediately.

Let $K = G \times J$ be a right group and T a left simple semigroup such that $K \cap T = \emptyset$. Let $\eta : t \to \eta_t$ be a mapping of T into the semigroup of all epimorphisms of J onto itself and let $\phi : T \to G$ be a homomorphism. Let us define a multiplication on $\Sigma = G \times J \cup T$ by

$$(x,i)(y,j) = (xy,j) \qquad t(x,i) = (\phi(t)x,i)$$
$$(x,i)t = (x\phi(t), i\eta_t) \qquad ts = r \in T, \text{ then } ts = r \in \Sigma$$

for each $i,j \in J$; $x,y \in G$; $t,s \in T$. Then Σ , with a multiplication defined in this way, is a semigroup to be denoted by $M_3(G;J;T,\phi,\eta)$.

LEMMA 5.2. *A semigroup* S *contains a kernel* K *which is a right group,* $S \setminus K = T$ *is a left simple semigroup and* $K \subseteq Ka$ *for each* $a \in T$ *if and only if* S *is isomorphic with some* $M_3(G;J;T,\phi,\eta)$.

P r o o f. Similar to the proof of Lemma 3.4.

Let K be a group, b a fixed element of K and $a \notin K$. Let us define a multiplication * on $\Sigma = K \cup \{a\}$ by

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for each $i,j \in I$; $x,y \in G$. Then, Σ with a miltiplication defined in this way is a semigroup, to be denoted by $M_2(G;I;a,b,\xi_a)$.

LEMMA 3.6. |2| . *A semigroup* S *contains a kernel* K *which is a left group,* $S \setminus K = \{a\}$ *and* $a^2 \in K$ *if and only if* S *is isomorphic with some* $M_2(G;I;a,b,\xi_a)$.

By Lemmas 3.4, 3.5 and 3.6. we have

THEOREM 3.1. *Every proper left ideal of a semigroup* S *is a completely simple semigroup if and only if one of the following conditions hold:*

1° $S \cong M(G;I,J;P)$

2° $S \cong M_1(G;I,J;P;T,\phi,\psi,\xi,\eta)$

3° $S \cong M_2(G;I;a,b,\xi_a)$.

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P r o o f. (i) => (ii). Let a left ideal L of the semigroup S be a right group. Then L contains a left ideal G which is a group, so $SG = SGG \subset S\,LG \subset LG \subset G$. Thus G is a left ideal of S.

(ii) => (iii). See Corollary 2. |1|.

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5. SEMIGROUPS IN WHICH EVERY PROPER LEFT IDEAL IS A RIGHT RIGHT GROUP

LEMMA 5.1. *Every proper left ideal of a semigroup* S *is a right group if and only if one the following conditions hold:*

1^{o} S *has a kernel* K *which is a right group and* $S \cap K = \emptyset$ *or* $S \setminus K = T$ *is a left simple semigroup and* $K \subseteq Ka$ *for each* $a \in T$;

2^{o} S *has a kernel* K *which is a group and* $S \setminus K = \{a\}$, $a^2 \in K$.

P r o o f. If every proper left ideal of S is a right group, then by Lemma 3.4. there are three cases. Let the kernel K of S be a left group, $S \setminus K = \{a\}$, $a^2 \in K$. K has to be a right group also. Thus K is a group. The other cases and the converse follow immediately.

Let $K = G \times J$ be a right group and T a left simple semigroup such that $K \cap T = \emptyset$. Let $\eta : t \to \eta_t$ be a mapping of T into the semigroup of all epimorphisms of J onto itself and let $\phi : T \to G$ be a homomorphism. Let us define a multiplication on $\Sigma = G \times J \cup T$ by

$$(x,i)(y,j) = (xy,j) \qquad t(x,i) = (\phi(t)x,i)$$
$$(x,i)t = (x\phi(t), i\eta_t) \qquad ts = r \in T, \text{ then } ts = r \in \Sigma$$

for each $i,j \in J$; $x,y \in G$; $t,s \in T$. Then Σ , with a multiplication defined in this way, is a semigroup to be denoted by $M_3(G;J;T,\phi,\eta)$.

LEMMA 5.2. *A semigroup* S *contains a kernel* K *which is a right group,* $S \setminus K = T$ *is a left simple semigroup and* $K \subseteq Ka$ *for each* $a \in T$ *if and only if* S *is isomorphic with some* $M_3(G;J;T,\phi,\eta)$.

P r o o f. Similar to the proof of Lemma 3.4.

Let K be a group, b a fixed element of K and $a \notin K$. Let us define a multiplication * on $\Sigma = K \cup \{a\}$ by

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$x*y = xy$, $x,y \in K$; $\quad a*x = bx$, $x \in K$;
$x*a = xb$, $x \in K$; $\quad a*x = b^2$.

Then, Σ with this multiplication is a semigroup, to be denoted by $M_4(K;a,b)$.

It is easy to prove the following

LEMMA 5.3. *A semigroup* S *has a kernel* K *which is a group and* $S \setminus K = \{a\}$, $a^2 \in K$ *if and only if* S *is isomorphic with some* $M_4(K;a,b)$.

THEOREMA 5.1. *Every proper left ideal of a semigroup* S *is a right group if and only if one of the following condition hold :*

1^o $S \cong M_3(G;J;T,\phi,\eta)$, *(T can be empty)*
2^o $S \cong M_4(K;a,b)$.

P r o o f. Lemmas 5.1, 5.2. and 5.3. imply the assertion.

REMARK. Semigroups in which every proper left ideal is a left group are described by S.Bogdanović, |2|.

6. SEMIGROUPS IN WHICH EVERY PROPER BI-IDEAL IS A COMPLETELY SIMPLE SEMIGROUP

Let $K = M(G;I,J;P)$ be a Rees matrix semigroup and let T be a group such that $K \cap T = \emptyset$.

Let $\xi : p \to \xi_p$ be a mapping of T into the semigroup of all sirjection of I onto itself, and let $\eta : p \to \eta_p$ be a mapping of T into the semigroup of all sirjections of J onto itself, and let the conditions (i), from construction of Lemma 3.5., hold. Iet mappings $\phi : T \times I \to G$ and $\psi : T \times J \to G$ fulfil the conditions (ii)-(iv). We can define a multiplication on $\Sigma = K \cup T$ by (1)-(4). Then Σ with such a defined multiplication is a semigroup. Let us denote it by $M_5(G;I,J;P;T,\phi,\psi,\xi,\eta)$.

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LEMMA 6.1. *A semigroup* S *has a kernel* K *which is a completely simple semigroup,* $S \setminus K$ *is a group and for each* $a \in S \setminus K$, $K \subset Ka \cap aK$ *if and only if* S *is isomorphic with some* $M_5(G;I,J;P;T,\phi,\psi,\xi,\eta)$.

THEOREM 6.1. *Every proper bi-ideal of a semigroup* S *is a completely simple semigroup if and only if one of the following conditions hold:*

1^o $S \cong M(G;I,J;P)$

2^o $S \cong M_5(G;I,J;P;T,\phi,\psi,\xi,\eta)$

3^o $S \cong M_4(K;a,b)$.

P r o o f. Let every proper bi-ideal of S be a completely simple semigroup. Then Lemma 3.4. S has a kernel K which is a completely simple semigroup and $S \setminus K$ is a left simple semigroup and $K \subseteq Ka$ for each $a \in S \setminus K$. By Lemma which is dual to the Lemma 3.4. $S \setminus K$ is, also, right simple and $K \subseteq aK$ for each $a \in S \setminus K$. Thus $S \setminus K$ is a group and $K \subseteq aK \cap Ka$ for each $a \in S \setminus K$. If K is a left group, $S \setminus K = \{a\}$, $a^2 \in K$, then K has to be a right group and so K is a group and the assertion follows by Lemmas 6.1. and 5.3.

The converse follows immediately.

COROLLARY 6.1. *Every proper bi-ideal of* S *is a group if and only if one of the following conditions hold:*

1^o $S \cong M(G;I,J;P)$ *and* $|I| = 2$, $|J| = 1$ *or* $|I| = 1$, $|J| = 2$;

2^o $S \cong G \cup T$, $G \cap T = \emptyset$, G *and* T *are groups and the multiplication* $*$ *on* $G \cup T$ *is defined by using a homomorphism* $\phi : T \to G$ *in the following way*

$g*t = g\phi(t)$, $g \in G$, $t \in T$

$t*g = \phi(t)g$, $t \in T$, $g \in G$

$x*y = xy$ *in other cases,*

3^o $S \cong M_4(K;a,b)$

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REMARK. By Theorem 1. |17| and Corollary 6.1. we have that every proper bi-ideal of S is a group if and only if S is an F-semigroup, |17| , i.e. semigroup in which every proper left and right subideal of S is a group.

7. SEMIGROUPS IN WHICH ALL PROPER SUBSEMIGROUPS ARE SIMPLE

The following theorem is given in |4|. Let M(2,r) denote a monogenic semigroup with the index 2 and period r.

THEOREM 7.1. *Every proper subsemigroup of a semigroup S is simple if and only if one of the following conditions hold:*

1^{o} S *is* M(2,r) ;

2^{o} $|S| = 2$;

3^{o} S *is a completely simple periodic semigroup.*

P r o o f. Let every proper subsemigroup of S be simple, then S is a GE-semigroup (Theorem 2.3. |3|), also S is an M(2,r) or S is the union of periodic groups.

Let S be the union of periodic groups and $|S| > 2$.

If S is simple, then S is a completely simple periodic semigroup.

If S is not simple, then S has a completely simple kernel K, so (by Lemma 3.2) $S \setminus K = P$ is a left simple subsemigroup of S. Take idempotents, $e \in K$ and $f \in P$. They generated a semigroup T,

$$T = \langle e,f \rangle = \{e,f\} \cup \langle ef \rangle \cup \langle fe \rangle \cup \langle efe \rangle \cup \langle fef \rangle .$$

If $T \neq S$, then T is simple, thus

$$T = \langle ef \rangle \cup \langle fe \rangle \cup \langle efe \rangle \cup \langle fef \rangle \subset K.$$

i.e. $f \in K$. Contradiction. If $T = S$, then

$$T = \langle ef \rangle \cup \langle fe \rangle \cup \langle efe \rangle \cup \langle fef \rangle \subset K.$$

Contradiction.

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Conversely, take case 3^{o}, i.e. let S be a completely simple periodic semigroup and S´ a subsemigroup of S. Then S = = M(G;I,J;P), G is a periodic group. Let $H_{ij} = \{(g;i,j): g \in G\}$. Changing the index, if necessary, we can get S´ and H_{11} to have a nonempty interesection. G´ = $\{g \in G:(g:1,1) \in S´\}$ is a subsemigroup of the periodic group G, thus, it is a subgroup of G. Immediately we can prove that $S \cap H_{ij} \neq \emptyset$ implies $S´ \cap H_{il} \neq \emptyset$.

In that case $S´ \cap H_{il} = \{(g;i,1):g \in G´\}$ and $S´ \cap H_{ij} =$ $= \{(g;i,j):g \in G\}$ are provable. Also, if $S´ \cap H_{ij} \neq \emptyset$ then the idempotent $(p_{ji}^{-1};i,j)$ is in S´ and p_{ji} is in G´. Finally say I´ = $= \{i \in I:S´ \cap H_{il} \neq \emptyset\}$ and J´ = $\{j \in J:S´ \cap H_{ij} \neq \emptyset\}$ and therefore S´ = = M(G´;I´,J´;P´).

Cases 1^{o} and 2^{o} follow immediately.

REMARK. Theorem 7.1. was given in |4|, but the proof was not completely correct. This is the reason wy it is given again with its complete proof. This theorem is a generalization of Proposition 1.1. |12| and the second part of the proof of the Theorem 7.1. is the same as the proof of Proposition 1.1. |12|.

COROLLARY 7.1. |15| . *Every proper subsemigroup of S is a group if and only if one of the following conditions hold:*

1^{o} S *is* M(2,r);
2^{o} |S| = 2 ;
3^{o} S *is a periodic group.*

8. SOME ISOMORPHISM THEOREMS

THEOREM 8.1. *Two semigroups* $S = M_1(G;I,J;P;T,\phi,\psi,\xi,\eta)$ *and* $S^* = M_1(G^*;I^*,J^*;P^*;T^*,\phi^*,\psi^*,\xi^*,\eta^*)$ *are isomorphic if and only if there is an isomorphism* $\omega : G \to G^*$, *a mapping* $i \mapsto u_i$ *of* I *into* G*, a mapping* $j \mapsto v_j$ *of* J *into* G*, bijections* $h : I \to I^*$ *and* $k : J \to J^*$ *and an isomorphism* $\Omega : T \to T^*$ *such that*

1) $p_{ji}\omega = v_j p^*_{jkih} u_i$
2) $\xi_p h = h\, \xi^*_{p\Omega}$
3) $\eta_p k = k \eta^*_{p\Omega}$
4) $\phi(p,i)\omega = u^{-1}_{i\xi_p}\, \phi^*(p\Omega,ih)u_i$

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5) $\psi(p,j)\omega = v_j \psi^*(p\Omega, jk) v^{-1}_{j\eta_p}$

COROLLARY 8.1. *Two semigroups* $M_3(G;J;T,\phi,\eta)$ *and* $M_3(G^*;J^*;T^*,\phi^*,\eta^*)$ *are isomorphic if and only if there exists an isomorphism* $\omega : G \to G^*$, *a bijection* $k : J \to J^*$ *and an isomorphism* $\Omega : T \to T^*$ *such that* $\phi\omega = \Omega\phi^*$ *and* $\eta_p k = k\eta^*_{p\Omega}$.

THEOREM 8.2. *Two semigroups* $M_2(G;I;a,b,\xi_a)$ *and* $M_2(G^*;I^*;a^*,b^*,\xi_{a^*})$ *are isomorphic if and only if there is an isomorphism* $\omega : G \to G^*$ *and a bijection* $h : I \to I^*$ *such that* $b\omega = b^*$ *and* $\xi_a h = h\xi^*_{a^*}$.

COROLLARY 8.2. *Two semigroups* $M_4(G;a,b)$ *and* $M_4(G^*;a^*,b^*)$ *are isomorphic if and only if there is an isomorphism* $\omega : G \to G^*$ *such that* $\omega(b) = b^*$.

We have given some isomorphism theorems but we shall only prove Theorem 8.1. because the other proofs are similar to this one.

P r o o f of Theorem 8.1. Let $f : S \to S^*$ be an isomorphism $G \times I \times J$ is a kernel in S and $G^* \times I^* \times J^*$ is a kernel in S^*, therefore $G \times I \times J \cong G^* \times I^* \times J^*$. Now, by Theorem 2.8. |10| , there exists an isomorphism $\omega : G \to G^*$, bijections $h : I \to I^*$, $k : J \to J^*$ and elements $u_i, v_j \in G$ $(i \in I,\ j \in J)$ such that condition 1) is satisfied. For $p \in T$ and $(a;i,j) \in G \times I \times J$ we have

$$(p(a;i,j))f = (\phi(p,i)a; i\xi_p, j)f =$$

$$= [u_{i\xi_p} (\phi(p,i)a)\omega v_j; i\xi_p h, jk] \in G^* \times I^* \times J^*$$

and

$$(pf)((a;i,j)f) = p\Omega[u_i(a\omega)v_j; ih, jk] =$$

$$= [\phi^*(p\Omega, ih) u_i(a\omega) v_j; ih\xi^*_{p\Omega}, jk]$$

so $\xi_p h = h\xi^*_{p\Omega}$, i.e. condition 2) holds. The proof of 3) is similar. Further,

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$$u_{i\xi_p}(\phi(p,i)a)\omega v_j = \phi^*(p\Omega,ih)u_i(a\omega)v_j$$

therefore

$$\phi(p,i)\omega = u^{-1}_{i\xi_p}\phi^*(p\Omega,ih)u_i .$$

Hence, condition 4) is satisfied. The proof of 5) is similar.

Conversely, let us define a mapping $f : S \to S^*$ by

$$xf = \begin{cases} [u_i(a\omega)v_j; ih, jk] , & x \in G \times I \times J \\ x\Omega , & x \in T . \end{cases}$$

f is an isomorphism. For $x,y \in G \times I \times J$ f is an isomorphism by Theorem 2.8. |10|. Ω is an isomorphism of T onto T* . It remains to prove that f is an isomorphism for $p \in T$ and $(a;i,j) \in G \times I \times J$. Indeed,

$$(p(a;i,j))f = (\phi(p,i)a; i\xi_p, j)f =$$

$$= [u_{i\xi_p}(\phi(p,i)a)\omega v_j; i\xi_p h , jk] =$$

$$= [u_{i\xi_p}\phi(p,i)\omega(a\omega)v_j; ih\xi^*_{p\Omega}, jk] =$$

$$= [u_{i\xi_p}u^{-1}_{i\xi_p}\phi^*(p\Omega,ih)u_i(a\omega)v_j; ih\xi^*_p, jk] =$$

$$= [\phi^*(p\Omega,ih)u_i(a\omega)v_j; ih\xi^*_{p\Omega}, jk]$$

and

$$(pf)((a;i,j)f) = p\Omega[u_i(a\omega)v_j; ih, jk] =$$

$$= [\phi^*(p\Omega,ih)u_i(a\omega)v_j; ih\xi^*_{p\Omega}, jk] :$$

Thus $(p(a;i,j))f = (pf)((a;i,j)f)$. The proof of $((a;i,j)p)f = ((a;i,j)f)(pf)$ is similar. Thus, f is as isomorphism.

REMARK. If T and T* are partial semigroups, then a similar Theorem to Theorem 8.1. can be given. In that case instead of the isomorphism $\Omega : T \to T^*$ we have a partial isomorphism. Such a theorem (without restriction for η_p) is given without a proof in |16|.

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REZIME

POLUGRUPE SA POTPUNO PROSTIM JEZGROM

U ovom radu daju se neke nove karakterizacije pomoću bi-ideala i levih ideala za polugrupe sa potpuno prostim jezgrom Takodje, u ovom radu razmatraju se polugrupe u kojima svaki pravi levi ideal (bi-ideal) jeste potpuno prosta polugrupa i teoreme o izomorfizmu ovakvih polugrupa su date.

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FUZZY CONGRUENCE RELATIONS AND CONSTRUCTIONS OF ALGEBRAS

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ABSTRACT

The Boolean extension of an algebra (|1|) is here given as a collection of fuzzy sets. It is shown that fuzzy relations, especially fuzzy congruence relations (|2|,|3|) are closely related to those structures. Some applications of fuzzy relations in constructing new algebras and extensions are given.

1. A fuzzy binary relation ρ on the set A, is a fuzzy subset of $A \times A$, with the membership function $\mu_\rho : A \times A \to B$, where $B = \langle B, \wedge, \vee, ', 0, 1\rangle$ is a Boolean algebra.

2. A fuzzy equivalence relation ρ on A is a fuzzy binary relation on A, which satisfies:

a) $(\forall a \in A)(\mu_\rho(a,a) = 1)$ (ρ is reflexive);

b) $(\forall a,b \in A)(\mu_\rho(a,b) = \mu_\rho(b,a))$ (ρ is symmetric);

c) $\mu_\rho(a,b) \geq \bigvee_{c \in A} (\mu_\rho(a,c) \wedge \mu_\rho(c,b))$, for all $a,b \in A$. (ρ is transitive).

3. If $A = \langle A, O\rangle$ is an algebra then a fuzzy binary relation θ on A is a fuzzy congruence relation on A, iff it is a fuzzy equivalence relation on A, satisfying the substitution property:

If $\mu_\theta(a_i,b_i) = p_i$, $i = 1,\ldots,n$, $(p_i \in B)$, then for every n-ary operation $f \in O$,

$$(f(a_1,\ldots,a_n), f(b_1,\ldots,b_n)) \geq \bigwedge_{i=1}^{n} p_i \quad (|2|).$$

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DEFINITION 1.[1] *Let* $A = \langle A, O \rangle$ *be an algebra, and* B *the complete Boolean algebra. The collection* $A(B)$ *of fuzzy sets* X *on* A *is said to be a Boolean extension of* A *iff for every* $X \in A(B)$:

1. $a \neq b$, $a,b \in A$, *implies:* $\mu_X(a) \wedge \mu_X(b) = 0$, *and*

2. $\bigvee_{a \in A} \mu_X(a) = 1$.

$A(B) = \langle A(B), O \rangle$ is a new algebra, with the operations defined in the following way:

$f(X_1, X_2, \ldots, X_n) = Y$ where, for $a \in A$,

$\mu_Y(a) = \bigvee_{f(a_1,\ldots,a_n)=a} (\mu_{X_1}(a_1) \wedge \ldots \wedge \mu_{X_n}(a_n))$, and the supremum runs over all n-tuples $(a_1, \ldots, a_n)$, such that $f(a_1, \ldots, a_n) = a$.

It is easy to check that the operations are well defined, i.e. that Y is also a fuzzy set satisfying 1) and 2). The supremum used in defining the operations explains why B has to be complete (provided that A is not finite).

REMARK

If $B \simeq P(I)$ (as the Boolean algebra), then $A(B)$ is isomorphic to the direct power A^I([4]).

In the following we shall consider some special fuzzy relations on $A(B)$.

PROPOSITION 1. *Let* A *be an algebra, and* B *the complete Boolean algebra. Let* θ *be an (ordinary) congruence relation* A. *Then the fuzzy relation* $\underline{\theta}$, *given by*

$$\mu_{\underline{\theta}}(X,Y) = \bigvee_{(a,b) \in \theta} (\mu_X(a) \wedge \mu_Y(b)), \quad X,Y \in A(B),$$

is a fuzzy congruence relation on $A(B)$.

P r o o f. $\underline{\theta}$ is a fuzzy equivalence relation on $A(B)$: It is obvious that $\underline{\theta}$ is reflexive and symmetric in the sense of a) and b) in 2. We shall here prove that it is transitive:

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$$\mu_{\underline{\theta}}(X,Z) \wedge \mu_{\underline{\theta}}(Z,Y) =$$

$$= \bigvee_{(a,c_1)\in\theta} (\mu_X(a) \wedge \mu_Z(c_1)) \wedge \bigvee_{(c_2,b)\in\theta} (\mu_Z(c_2) \wedge \mu_Y(b)) =$$

$$= \bigvee (\mu_X(a) \wedge \mu_Z(c_1) \wedge \mu_Z(c_2) \wedge \mu_Y(b); (a,c_1)\in\theta, (c_2,b)\in\theta) =$$

$$= \bigvee (\mu_X(a) \wedge \mu_Z(c) \wedge \mu_Y(b); (a,c)\in\theta, (c,b)\in\theta) \leq$$

$$\bigvee_{(a,b)\in\theta} (\mu_X(a) \wedge \mu_Y(b)) = \mu_{\underline{\theta}}(X,Y),$$

by 1), Definition 1., and since θ is transitive. Now let $\mu_{\underline{\theta}}(X_i,Y_i) = r_i$, $i=1,\ldots,n$, and for an n-ary operation f, $\mu_{\underline{\theta}}(f(X_1,\ldots,X_n), f(Y_1,\ldots,Y_n)) = r$. Then,

$$\bigwedge_{i=1}^{n} r_i = \bigwedge_{i=1}^{n} \left(\bigvee_{(a,b)\in\theta} (\mu_{X_i}(a) \wedge \mu_{Y_i}(b)) \right) =$$

$$\bigvee \left(\bigwedge_{i=1}^{n} (\mu_{X_i}(a_i) \wedge \mu_{Y_i}(b_i)); (a_i,b_i)\in\theta, i=1,\ldots,n \right) \leq$$

$$\bigvee \left(\bigwedge_{i=1}^{n} (\mu_{X_i}(a_i) \wedge \mu_{Y_i}(b_i)); (f(a_1,\ldots,a_n), f(b_1,\ldots,b_n))\in\theta \right) =$$

$$\bigvee_{(a,b)\in\theta} \left(\bigvee \left(\bigwedge_{i=1}^{n} \mu_{X_i}(a_i); f(a_1,\ldots,a_n)=a \right) \wedge \bigvee \left(\bigwedge_{i=1}^{n} \mu_{Y_i}(b_i); f(b_1,\ldots,b_n)=b \right) \right) = \mu_{\underline{\theta}}(f(X_1,\ldots,X_n), f(Y_1,\ldots,Y_n)),$$

which proves the substitution property.

If θ is the equality on A, then the corresponding fuzzy congruence relation on $A(B)$ can be given as follows:

COROLLARY 2. *Fuzzy relation* $\underline{\sigma}$ *on* $A(B)$, *given by its membership function*

$$\mu_{\underline{\sigma}}(X,Y) = \bigvee_{a\in A} (\mu_X(a) \wedge \mu_Y(a)),$$

is a fuzzy congruence relation on $A(B)$.

* * *

In the following propositions we shall describe the use of fuzzy congruence relations in some constructions of algebras, based on Boolean extensions.

DEFINITION 2. *Let* $A(B)$ *be the Boolean extension of* A, $\underline{\sigma}$ *the fuzzy congruence relation on* $A(B)$ *(defined in Corollary 2.)*

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and F *any filter in* B. *The reduced extension* $A(B)_F$ *of* A *is an algebra in which*

$$A(B)_F = \{\sigma(X_F) \; ; \quad X \in A(B)\} \quad , \text{ and}$$

$$\sigma(X)_F = \{Y; Y \in A(B) \qquad \text{and} \qquad \mu_{\underline{\sigma}}(X,Y) \in F\}.$$

The operations on these collections of fuzzy sets are defined in the natural way:

$$f(\sigma(X_1)_F, \ldots, \sigma(X_n)_F) \overset{\text{def}}{=\!=} \sigma(f(X_1, \ldots, X_n))_F \; ,$$

and it is easy to show that the resulting element does not depend on the representatives, the fuzzy sets $X_1, \ldots, X_n$.

Clearly, the following lemma holds.

LEMMA 3. *If* F *is a principal ultrafilter in* B, *then*

$$A(B)_F \cong A \; .$$

P r o o f. The isomorphism is given by the mapping $h : A \to A(B)_F$, where $h(a) = \sigma(X)_F$ and $\mu_X(a) = 1$.

Let p and q be two arbitrary elements of a Boolean algebra B. Denote the Boolean expression $(p \vee q') \wedge (p' \vee q)$ by $p \Theta q$.

PROPOSITION 4. *Let* $A = \langle\{a,b\}, 0\rangle$ *be a two-element algebra, and* B *an arbitrary Boolean algebra. Then there is a* 1 - 1 *map of* B *into* $A(B)(p \to X_p)$ *such that if* $\underline{\sigma}$ *is a fuzzy congruence relation defined in Corollary 2, then for all* $p,q \in B$, $\mu_{\underline{\sigma}}(X_p, X_q) = p \Theta q$.

P r o o f. If $X \in A(B)$, then $\mu_X(a) = p$ implies $\mu_X(b) = p'$. Hence, every $p \in B$ uniquely determines one X, and if $p \neq q$, then clearly $X_p \neq X_q$. Now, $\mu_{\underline{\sigma}}(X_p, X_q) = (\mu_{X_p}(a) \wedge \mu_{X_q}(a)) \vee (\mu_{X_p}(b) \wedge \mu_{X_q}(b)) = (p \wedge q) \vee (p' \wedge q') = (p \wedge q') \wedge (p' \wedge q) = p \Theta q$.

Now let $p^k = \begin{cases} p', & \text{if } k = 0 \\ p, & \text{if } k = 1 \end{cases}$, $p \in B$, $k \in \{0,1\}$.

A converse of the preceeding proposition can be formulated as follows.

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PROPOSITION 5. *Let* $A = \langle A, \sigma \rangle$ *be an algebra,* B *any Boolean algebra, and suppose that there is and* $1-1$ *map* x_p *of* B *onto* A, *and a fuzzy congruence relation* $\underline{\sigma}$ *on* A, *with* $\mu_\sigma(x_p, x_q) = p \Theta q$. *Also let* $f(x_{p_1}, \ldots, x_{p_n}) = x_q$, $(f \in O)$, *where*

(a) $$q = \bigvee (p_1^{i_1} \wedge \ldots \wedge p_n^{i_n}) \; ; \; f(X_{F^{i_1}}, \ldots, X_{F^{i_n}}) = X_F)$$

$$A/_{\sigma_F} = \langle \{X_F, X_{F'}\}, O \rangle$$

and X_F *is a class to which* X_1 $(1 \in B)$ *belongs.*

Then there is a two element algebra A_1, *such that*

$$A_1(B) \cong A \; .$$

P r o o f. Let $A_1 = A/_{\sigma_F}$, and denote the elements of $A_1(B)$ by X_p, if $\mu_{X_p}(X_F) = p$. The map h of $A_1(B)$ onto A: $h(X_p) = x_p$, is an isomorphism. Really, it is $1-1$ by definition. Also,

$$h\,(f(X_{p_1}, \ldots, X_{p_n})) = h(X_q), \quad \text{where,}$$

$$\mu_{X_q}(X_F) = \bigvee (\mu_{X_{p_1}}(X_{F^{i_1}}) \wedge \ldots \wedge \mu_{X_{p_n}}(X_{F^{i_n}}) \; ;$$

$$f(X_{F^{i_1}}, \ldots, X_{F^{i_n}}) = X_F) = \bigvee (p_1^{i_1} \wedge \ldots \wedge p_n^{i_n} \; ;$$

$$f(X_{F^{i_1}}, \ldots, X_{F^{i_n}}) = X_F, \quad \text{and thus,}$$

by (a), $h(X_q) = x_q = f(x_{p_1}, \ldots, x_{p_n}) = f(h(X_{p_1}), \ldots, h(X_{p_n}))$.

We shall describe now some fuzzy properties of induced notions, and then we shall use them in describing one construction of algebras. First, we repeat some definitions from [3].

i) A fuzzy function from A to A_1 is a fuzzy relation $\underline{f}$ from A to A_1 such that

- for every $a \in A$, there is exactly one $X \in A_1$, such that $\mu_{\underline{f}}(a, X) = 1$;

1) σ_F is a regular binary relation: $(a,b) \in \sigma_F$ iff $\mu_\sigma(a,b) \in F$ (see [4]).

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- every $p \in B$, $p \neq 0$, appears once at most as a value of $\mu_{\underline{f}}(a,X)$, $X \in A_1$.

ii) A fuzzy function $\underline{h}$ from the algebra $A_1 = \langle A_1, 0\rangle$ to the algebra $A_2 = \langle A_2, 0\rangle$ is said to be a fuzzy homomorphism from A_1 to A_2, if from

$$\mu_{\underline{h}}(a_i,b_i) = p_i,\ a_i \in A_1, b_i \in A_2,\ i=1,\ldots,n, \text{ it follows that}$$

$$\mu_{\underline{h}}(f(a_1,\ldots,a_n),f(b_1,\ldots,b_n)) \in ([\bigwedge_{i=1}^{n} p_i)^*,\ f \in 0\ .$$

LEMMA 6. *Let* $X \in A(B)$, $p \in B$, *and* $\underline{\sigma}$ *the fuzzy congruence relation defined in Corollary 2. Then there is* $\underline{Y} \in A(B)$ *such that* $\mu_{\underline{\sigma}}(\underline{X},\underline{Y}) = p$.

P r o o f. Let a and b be two arbitrary elements from algebra A, and let $\mu_{\underline{X}}(a) = p_a$, $\mu_{\underline{X}}(b) = p_b$. Define $\underline{Y} \in A(B)$ as follows:

$$\mu_{\underline{Y}}(a) = (p \wedge p_a) \vee \bigvee_{x \neq a} (p' \wedge p_x),\ (x \in A),\ \mu_{\underline{Y}}(b) =$$

$$= (p \wedge p_b) \vee (p' \wedge p_a)\ , \text{ and}$$

$$\mu_{\underline{Y}}(x) = p \wedge p_x,\ x \neq a,\ x \neq b.$$

Now it is easy to show that $\underline{Y} \in A(B)$, and that

$$\mu_{\underline{\sigma}}(\underline{X},\underline{Y}) = p.$$

Now, starting with $A(B)$, we can construct a new algebra $A_{\underline{\sigma}}$, by means of the fuzzy congruence relation $\underline{\sigma}$ (generalizing the notion of a quotient algebra). Let

$$A_{\underline{\sigma}} = \bigcup_{p \in B} A(B)_{\lceil p)}\ .$$

The operations are defined as in |3|:

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*) $\lceil p)$ is a principal filter in B, generated by p.

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$$f(\sigma(a_1)\lceil p), \ldots, \sigma(a_n)\lceil p)) = \sigma(a\ \lfloor \bigwedge_{i=1}^{n} p_i),$$

$a = f(a_1, \ldots, a_n)$, $f \in O$.

The fact that p runs over all elements of B, is the consequence of Lemma 6.

PROPOSITION 7. *Algebra* $A_{\underline{\sigma}}$ *is a homomorphic image of* $A(B)$.

P r o o f. The fuzzy relation $\underline{h}$ from $A(B)$ to $A_{\underline{\sigma}}$, defined as

$$\mu_{\underline{h}}(\underline{x}, X) = \begin{cases} V(p_i,\ \sigma(\underline{x})\lceil p_i) = X), & \text{if } \underline{x} \in X \\ 0, & \text{otherwise,} \end{cases}$$

is the required fuzzy homomorphism, as in |3|, and by Lemma 6.

If $A(B)$ is the extension of a two-element algebra A, then by Lemma 6, and by Proposition 4, every $p \in B$ appears exactly ones as a value $\mu_{\underline{\sigma}}(a, \underline{x})$, for an $\underline{a} \in A(B)$. Consider now the fuzzy relation $\underline{g}$ from A to $A(B)$:

$$\mu_{\underline{g}}(a, \underline{X}) = \mu_{\underline{\sigma}}(\underline{P}, \underline{X}), \text{ where } \underline{P} \in A(B), \text{ and } \mu_{\underline{P}}(a) = 1.$$

PROPOSITION 8. *A fuzzy relation* $\underline{g}$ *is a fuzzy homomorphism from a two-element algebra* A *to* $A(B)$.

P r o o f. By construction $\underline{g}$ is a fuzzy function. It satisfies the supstitution property:

Clearly, $\mu_{\underline{g}}(a, \underline{X}) = \mu_{\underline{X}}(a)$. Thus,

$$\mu_{\underline{\sigma}}(a_i, \underline{X}_i) = \mu_{\underline{X}_i}(a_i), \quad i=1, \ldots, n, \text{ and}$$

$$\mu_{\underline{\sigma}}(f(a_1, \ldots, a_n), f(\underline{X}_1, \ldots, \underline{X}_n)) = \mu_{\underline{Z}}(f(a_1, \ldots, a_n)),$$

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$\underline{Z} = f(\underline{X}_1,\ldots,\underline{X}_n)$. Now, since $\underline{Z}$ is from A(B),

$\mu_{\underline{Z}}(f(a_1,\ldots,a_n)) = V(\mu_{\underline{X}_1}(b_1) \wedge \ldots \wedge \mu_{\underline{X}_n}(b_n))$, for all $b_1,\ldots,b_n$ such that $f(a_1,\ldots,a_n) = f(b_1,\ldots,b_n)$, and this obviously is not less than $\mu_{\underline{X}_1}(a_1) \wedge \ldots \wedge \mu_{\underline{X}_n}(a_n)$.

We finally give one example of the fuzzy mappings from the Propositions 7. and 8., using an arbitrary Boolean algebra B:

$A = \langle\{a,b\},*\rangle$ $\quad B = \{0,1,p,p',q,q',\ldots\}$

*	a	b
a	a	b
b	b	a

$A(B) = \{\begin{pmatrix} a & b \\ 1 & 0 \end{pmatrix} = \underline{X}_1,\ \begin{pmatrix} a & b \\ 0 & 1 \end{pmatrix} = \underline{X}_o,\ \begin{pmatrix} a & b \\ p & p' \end{pmatrix} = \underline{X}_p, \ldots\}$.

$\underline{X}_p * \underline{X}_q = \begin{pmatrix} a & b \\ p \odot q & (p \odot q)' \end{pmatrix}$.

σ	$\underline{X}_o$	$\underline{X}_1$	$\underline{X}_p$	$\underline{X}_{p'}$	$\underline{X}_q$	...
$\underline{X}_o$	1	0	p'	p	q'	...
$\underline{X}_1$	0	1	p	p'	q	...
$\underline{X}_p$	p'	p	1	0	$p \odot q$	...
$\underline{X}_{p'}$	p	p'	0	1	$p' \odot q$	...
$\underline{X}_q$	q'	q	$p \odot q$	$p' \odot q$	1	...
.	.	.	.	.	.	.

$\underline{g}$	$\underline{X}_o$	$\underline{X}_1$	$\underline{X}_p$	$\underline{X}_{p'}$	$\underline{X}_q$	...
a	0	1	p	p'	q	...
b	1	0	p'	p	q'	...

$\underline{h}$	$\{\underline{X}_o\}$	$\{\underline{X}_1\}$	$\{\underline{X}_p\}$	$\{\underline{X}_o,\underline{X}_{p'},\ldots\}$	$\{\underline{X}_1,\underline{X}_p,\ldots\}$	...	$\{\underline{X}_o,\underline{X}_p,\ldots\}$	$\{\underline{X}_1,\underline{X}_{p'}\}$	...
$\underline{X}_o$	1	0	0 ...	p	0	...	p'	0	...
$\underline{X}_1$	0	1	0 ...	0	p	...	0	p'	...
$\underline{X}_p$	0	0	1 ...	0	p	...	p'	0	...
$\underline{X}_{p'}$	0	0	0 ...	p	0	...	0	p'	...

. . .

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REZIME

RASPLINUTE KONGRUENCIJE I KONSTRUKCIJE ALGEBRI

U radu se pokazuje da se koncept rasplinutih skupova može proširiti i na Bulove ekstenzije algebri i da se na njima mogu posmatrati rasplinute relacije. Opisana je jedna klasa rasplinutih relacija, kongruencija na tim algebrama i pokazano je da se one mogu koristiti za konstruisanje i opisivanje nekih konstrukcija algebri.

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